ELLIPTIK TENGLAMALAR UCHUN GALERKIN USULI YORDAMIDA VARIATSION -AYIRMALI YECHIM

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Annotatsiya: Ushbu maqolada elliptic turdagi chegaraviy masalalarni yechishda keng qo'llaniladigan Galyorkin usuli asosida variatsion ayirmali yondashuv bayon qilinadi.Elliptik tenglamaning kuchsiz qo'yilishi, funksiya fazolari sinov funksiyalar tanlovi hamda diskretlash jarayoni bosqichma bosqich keltiriladi.

Kalit so'zlar: Elliptik tenglamalar, Galyorkin usuli, variatsion qo'yilish, variatsion – ayirmali usul, chegaraviy masala.

Galyorkin usuli xususiy hosilali differnsial tenglamalarni taqribiy yechishga mo'ljallangan variatsion usullardan biri bo'lib, u og'irlik funksiyalari usuli sinfiga kiradi. Ushbu usul XX asr boshlarida rus matematigi B.G.Galyorkin tomonidan taklif qilingan bo'lib, differensial tenglamalarni taqribiy yechishda keng qo'llaniladi.

Galyorkin usulining asosiy g'oyasi shundan iboratki, differensial tenglamalarning aniq yechim o'rniga oldindan tanlangan Cheklangan o'lchamli fazoda taqribiy yechim izlanadi va qoldiq funksiya shu fazodagi funksiyalarga ortoganal qilinadi.

Galyorkin usulining g'oyasi. Rits usulining asosiy kamchiligi shundaki, u faqat operatori simmetrik va musbat bo'lgan tenglamalarga qo'llaniladi. Akademik B.G.Galyorkin 1915 yilda shunday usul taklif qildiki, u Rits usuliga nisbatan umumiydir. Bu usul xech qanday variatsion masala bilan bog'liq emas, shuning uchun ham u batamom universal usul hisoblanadi. Bu usulni elliptik, parabolik va giperbolik tenglamalarga, xatto ular variatsion masala bilan bog'liq bo'lmasa xam, katta muvaffaqiyat bilan qo'llash mumkin. Agar tenglamaning operatori simmetrik va musbat bo'lsa, Galyorkin usuli osonroq yo'l bilan Rits usuli beradigan taqribiy yechimni beradi. Taqribiy yechimning koeffitsiyentlarini aniqlaydigan chiziqli algebraik tenglamalar sistemasi bir hil bo'ladi. Galyorkin usulining yaqinlashishini akademik M.V. Keldish ko'rsatgan.

Endi Galyorkin usulining asosiy g'oyasi bilan tanishamiz. Faraz qilaylik,

$$Au = f(x,y) \tag{1}$$

tenglama berilgan bo‘lib, A- qandaydir ikki o‘zgaruvchili differensial operator bo‘lsin va (1) tenglamaning yechimi bir jinsli chegaraviy shartlarni qanoatlantirsin. Bu masalaning yechimini quyidagi ko‘rinishda izlaymiz:

$$u_n(x,y) = \sum_{k=1}^{n} a_k \psi_k(x,y) \ (2)$$

bu yerda $\psi_1(x,y), \psi_2(x,y), ..., \psi_n(x,y)$ funksiyalar berilgan G sohada to‘liq bo‘lgan chiziqli erkli $\psi_k(x)\ _{k=1}^{\infty}$ sistemaning avvalgi n tasi bo‘lib, bir jinsli chegaraviy shartlarni qanoatlantiradi. Taqribiy yechim $u_n(x,y)$ aniq yechimga aylanishi uchun $\varepsilon_n(x,y) = A\ \ u_n(x,y) - f(x,y)$ ifoda aynan nolga aylanishi kerak. Agar $\varepsilon_n(x,y)$ uzluksiz bo‘lsa, bu talab $\varepsilon_n(x,y)$ funksiya $\psi_k(x)\ _{k=1}^{\infty}$ sistemaning barcha funksiyalariga ortogonal bo‘lishi bilan teng kuchlidir. Ammo bizda faqat n ta $a_1, a_2, ..., a_n$ o‘zgarmaslar bo‘lganligi sababli ortogonallik shartining faqat n tasini qanoatlantira olamiz. Bu shartlar quyidagi tenglamalar sistemasiga olib keladi:

$$\iint_G \ A\ u_n(x,y)\ - f(x,y)\ \ \psi_j(x,y) dxdy = 0$$

yoki

$$\iint_G \ A\ u_n(x,y)\ \psi_j(x,y)\ \ dxdy = \iint_G f(x,y) \psi_j(x,y) dxdy$$

Ushbu a_i koeffitsiyentlarni topishga xizmat qiladi. Agar A operator chiziqli bo‘lsa, u holda bu sistema $a_1, a_2, ..., a_n$ larga nisbatan chiziqli algebraik tenglamalar sistemasidan iborat bo‘ladi. Bu sistemadan a_j larni topib (2) ga ko‘ysak, kerakli taqribiy yechimni xosil qilamiz.

Galyorkin metodida, yechimni yakunlash uchun asosiy funksiyalar (sinusoidal yoki polinomial funktsiyalar) tanlanadi va bu funksiya majmuasi orqali differensial tenglama yechiladi.

Agar berilgan masala variatsion shaklga ega bo‘lsa (elliptik tenglamalarda ko‘pincha shunday bo‘ladi), u holda Galyorkin usuli tabiiy ravishda variatsion formuladan kelib chiqadi.

Masalan, Puasson tenglamasi uchun:

$$\Delta u = f$$

variatsion ko‘rinish:

$$\int_{\Omega} \Delta u \Delta v d\Omega = \int_{\Omega} f v d\Omega$$

bu yerda v sinov funksiyasi bo‘lib, Galyorkin usulida $v = \varphi_j$ deb olinadi.

Galyorkin usulining algoritmi quyidagi bosqichlarda amalga oshiriladi:

1. Hisoblash sohasi tanlanadi,
2. Funksional fazo aniqlanadi,
3. Bazis funksiyalar tanlanadi,
4. Taqribiy yechim yoziladi,
5. Variatssion tenglama tuziladi,
6. Algebraic tenglamalar sistemasi hosil qilinadi,
7. Sistema yechilib, noma’lum koeffitsientlar aniqlanadi.

Galyorkin usuli rits usuli, Chegaralangan elementlar usuli, variatsion ayirmali usullar bilan uzviy bog’liq. Ayniqsa, Chegaralangan elementlar usuli galyorkin usuliga asoslanadi.

Misol.

Biz $\Omega = \{(x, y),\ 0 < x < 1,\ 0 < y < 1\}$ sohada $u_{xx} + u_{yy} = x^{2,}$ $u(x,0) = 0, u(0, y) = 0$ boshlang’ich va $u(x,1) = x$, $u(1, y) = y$ chegaraviy shartlarni qanoatlantiruvchi Puasson masalasini Galyorkin usuli yordamida variatsion – ayirmali yechimini ko’rib chiqamiz.

Biz masalamiz uchun bazis funksiyasini tanlaymiz,

$$\varphi_0 = xy$$

$$\varphi_{ij}(x, y) = c_{ij} x^i (1 - x) y^j (1 - y)$$

va taqribiy yechimni topamiz,

$$u_h(x,y)=c_{ij}\varphi_{ij}(x,y)=c_{ij}x^i(1-x)y^j(1-y)$$

Endi biz

$$\int_{\Omega}\Delta u(x,y)\varphi_{ij}(x,y)dxdy=\int_{\Omega}f(x,y)\varphi_{ij}(x,y)dxdy$$

Integralni yoyib chiqamiz,

$$\int_0^1\int_0^1 (u_{xx}+u_{yy})\,c_{ij}x(1-x)y(1-y)\,dxdy=\int_0^1\int_0^1 x^2c_{ij}x(1-x)y(1-y)\,dxdy$$

keying qiladigan ishimiz $i=\overline{1,2};\ j=\overline{1,2}$ larga son berib chiqamiz va 4 ta noma’lumli tenglamalar sistemasini hosil qilamiz, bu yerdan u_{xx} va u_{yy} larni topamiz,

$$u_{xx}=-2c_{ij}(y-y^2);\ \ u_{yy}=-2c_{ij}(x-x^2)$$

u_{xx} va u_{yy} larni topdik,

endi biz u_{xx} va u_{yy} larni quyidagi integral o’rniga qo’yamiz,

$$\begin{cases}\int_0^1\int_0^1(-2c_{11}(y-y^2)-2c_{11}(x-x^2))c_{11}x(1-x)y(1-y)\,dxdy=\int_0^1\int_0^1 x^2c_{11}x(1-x)y(1-y)\,dxdy\\ \int_0^1\int_0^1(-2c_{12}(y-y^2)-2c_{12}(x-x^2))c_{12}x(1-x)y^2(1-y)\,dxdy=\int_0^1\int_0^1 x^2c_{12}x(1-x)y^2(1-y)\,dxdy\\ \int_0^1\int_0^1(-2c_{21}(y-y^2)-2c_{21}(x-x^2))c_{21}x^2(1-x)y(1-y)\,dxdy=\int_0^1\int_0^1 x^2c_{21}x^2(1-x)y(1-y)\,dxdy\\ \int_0^1\int_0^1(-2c_{22}(y-y^2)-2c_{22}(x-x^2))c_{22}x^2(1-x)y^2(1-y)\,dxdy=\int_0^1\int_0^1 x^2c_{22}x^2(1-x)y^2(1-y)\,dxdy\end{cases}$$

integrallarni hisoblaymiz,

$$\begin{cases}-\dfrac{c_{11}^2}{45}=\dfrac{c_{11}}{120}\\ -\dfrac{c_{12}^2}{90}=\dfrac{c_{12}}{240}\\ -\dfrac{c_{21}^2}{90}=\dfrac{c_{21}}{180}\\ -\dfrac{c_{22}^2}{180}=\dfrac{c_{22}}{360}\end{cases}$$

tenglamalar sistemasidan noma’lumlarni topamiz,

$$\begin{cases} c_{11} = -\dfrac{3}{8} \\ c_{12} = -\dfrac{3}{8} \\ c_{21} = -\dfrac{1}{2} \\ c_{22} = -\dfrac{1}{2} \end{cases}$$

noma'lumlarni topib oldik endi umumiy yechimga qo'yamiz.

$$u_{ij}(x,y) = \varphi_0(x,y) + \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}\varphi_{ij}(x,y)$$

umumiy formula shu ko'rinishda bo'ladi o'rniga qo'yamiz,

$$u_{ij}(x,y) = xy + \sum_{0}^{1}\sum_{0}^{1} c_{ij}x^{i}(1-x)y^{j}(1-x)$$

c_{ij} larni o'rniga qo'yib maasalamizni umumiy ko'rinishini topamiz.

$$u_{ij} = xy - \frac{3}{8}x(1-x)y(1-y) - \frac{3}{8}x(1-x)y^2(1-y) - \frac{1}{2}x^2(1-x)y(1-y) - \frac{1}{2}x^2(1-x)y^2(1-y)$$

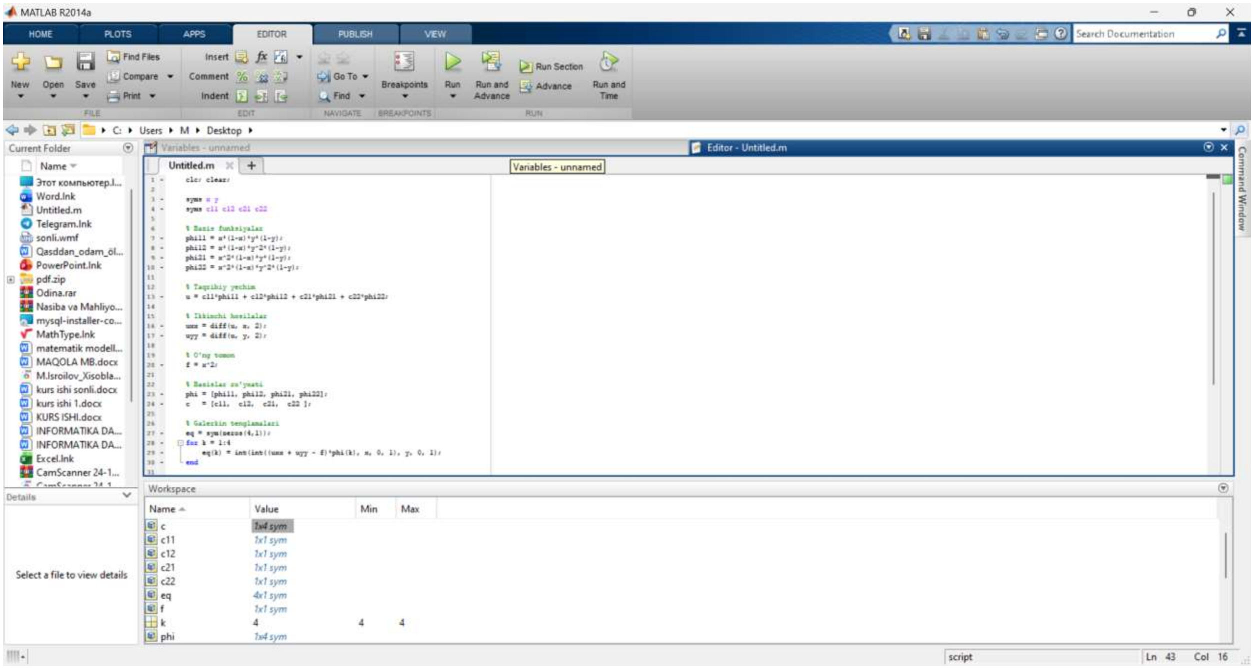
masalamiz yechildi.

Endi misolimizni dasturini tuzamiz.

```
clc; clear;
syms x y
syms c11 c12 c21 c22

phi11 = x*(1-x)*y*(1-y);      % Bazis funksiyalar
phi12 = x*(1-x)*y^2*(1-y);
phi21 = x^2*(1-x)*y*(1-y);
phi22 = x^2*(1-x)*y^2*(1-y);
u = c11*phi11 + c12*phi12 + c21*phi21 + c22*phi22;  % Taqribiy yechim
uxx = diff(u, x, 2);    % Ikkinchi hosilalar
uyy = diff(u, y, 2);
f = x^2;    % O'ng tomon
phi = [phi11, phi12, phi21, phi22];    % Bazislar ro'yxati
```

```
c   = [c11,  c12,  c21,  c22 ];
eq = sym(zeros(4,1));          % Galerkin tenglamalari
for k = 1:4
    eq(k) = int(int((uxx + uyy - f)*phi(k), x, 0, 1), y, 0, 1);
end
S = solve(eq == 0, c);            % Sistemani yechish
disp('Koeffitsientlar:')      % Natijalar
disp(S)
u_final = simplify(subs(u, c, [S.c11 S.c12 S.c21 S.c22]));      % Yakuniy yechim
disp('Taqribiy yechim u(x,y):')
pretty(u_final)
```



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