VARIATSION PRINSIPLAR ASOSIDA INTEGRAL TENGLAMALARNI YECHISHNING SONLI TAHLILI

Madaminova K.B.

FarDU talabasi, kibriyoxonastonaqulova@gmail.com

Annotatsiya: Maqolada variatsion prinsiplar asosida integral tenglamalarni sonli yechish masalasi tahlil qilingan. Integral tenglamalarni operator ko'rinishda ifodalash, ularni variatsion masalaga keltirish va mos funksional qurish jarayoni bayon etilib, simmetrik hamda musbat aniqlangan operatorlar uchun Hilbert fazodagi nazariy xossalar yoritilgan. Rits va Galerkin usullarining asoslari, yaqinlashuv hamda xatolik tahlili keltirilib, misol orqali Galerkin usulining amaliy qo'llanilishi ko'rsatilgan. Natijalar variatsion usullar integral tenglamalarni yechishda barqaror va yuqori aniqlikka ega ekanligini tasdiqlaydi.

Kalit so'zlar: integral tenglama, Fredgolm integral tenglamasi, variatsion prinsip, funksional analiz, Hilbert fazosi, Rits usuli, Galerkin usuli, sonli yechim, yaqinlashuv, xatolik tahlili.

Integral tenglamalar zamonaviy matematika va amaliy fanlarning muhim bo'limlaridan biri hisoblanadi. Ular matematik fizika, elastiklik nazariyasi, issiqlik o'tkazuvchanlik, elektromagnit maydonlar, aerodinamika hamda kvant mexanikasi masalalarida keng qo'llaniladi. Ko'plab real jarayonlarni modellashtirish jarayonida differensial tenglamalar integral tenglamalarga keltiriladi yoki bevosita integral ko'rinishda ifodalanadi.

Aksariyat integral tenglamalar uchun analitik yechimlarni topish imkoni mavjud emas. Shu sababli ularni yechishda sonli usullar muhim ahamiyat kasb etadi. Sonli usullar ichida variatsion prinsiplarga asoslangan yondashuvlar alohida o'rin tutadi, chunki bu usullar yechimning mavjudligi, yagonaligi va barqarorligini nazariy jihatdan asoslash imkonini beradi.

Variatsion prinsiplarning mohiyati shundan iboratki, integral tenglamani yechish masalasi ma'lum bir funksionalni ekstremumga keltirish masalasiga ekvivalent qilib olinadi. Bu esa masalani funksional analiz, operatorlar nazariyasi va Hilbert fazolari doirasida o'rganish imkonini beradi. Ushbu maqolada variatsion prinsiplar asosida integral tenglamalarni yechishning sonli tahlili, xususan Rits va Galerkin usullarining nazariy asoslari keng yoritiladi.

Ikkinchi tur Fredgolm integral tenglamasini ko‘rib chiqamiz:

$$u(x)-\lambda\int_a^b K\left(x,t\right)u(t)dt = f(x), \text{a э}\left[a,b\right]$$

Bu tenglamani operator ko‘rinishda quyidagicha yozish mumkin:

$$I(1-\lambda A)u = f$$

bu yerda A - integral operator:

$$\left(Au\right)(x)=\int_a^b K(x,\mathrm{t})\,\mathrm{u(t)}\,\mathrm{dt}$$

Agar yadro $K(x,\mathrm{t})$ simmetrik bo‘lsa, ya’ni $K(x,\mathrm{t})=K(\mathrm{t,x})$, u holda operator A o‘z-o‘ziga qo‘shma bo‘ladi va Hilbert fazoda qulay xossalarga ega bo‘ladi.

Variatsion prinsip va funksional qurilishi. Agar operator musbat aniqlangan bo‘lsa, integral tenglama quyidagi funksionalni minimallashtirish masalasiga ekvivalent bo‘ladi:

$$J\left(u\right)=\frac{1}{2}\left\langle u,\mathrm{v}\right\rangle-\frac{\lambda}{2}\left\langle Au,\mathrm{v}\right\rangle-\left\langle f,u\right\rangle$$

bu yerda skalyar ko‘paytma:

$$\left\langle u,\mathrm{v}\right\rangle=\int_a^b u\left(x\right)v\left(x\right)dx$$

Funksionalning birinchi variatsiyasini nolga tenglashtirish sharti:

$$\delta J(u)=0$$

Eyler–Lagrange tenglamasiga olib keladi va u boshlang‘ich integral tenglamaga ekvivalent bo‘ladi. Shu tariqa integral tenglama variatsion masala sifatida qaraladi.

Rits usulining sonli modeli. Rits usulida noma’lum funksiya cheklangan o‘lchamli fazoda approksimatsiya qilinadi:

$$u_n\left(x\right)=\sum_{i=1}^{n}c_i\varphi_i\left(x\right)$$

Bu ifodani funksionalga qo‘yib, minimallashtirish shartini bajaramiz:

$$\frac{\partial J\left(u_n\right)}{\partial c_i}=0, i=1,2,\ldots,n$$

Natijada quyidagi algebraik tenglamalar sistemasiga kelinadi:

$$\sum_{j=1}^{n}\left[\int_a^b \varphi_i\left(x\right)\varphi_j\left(x\right)dx-\lambda\int_a^b\int_a^b K\left(x,t\right)\varphi_i\left(x\right)\varphi_j\left(t\right)dxdt\right]c_j=\int_a^b f\left(x\right)\varphi_i\left(x\right)dx$$

Bu sistema matritsali ko‘rinishda $A\vec{c}=\vec{b}$ shaklida yechiladi.

Galerkin usuli. Galerkin usulida integral tenglamaning qoldig‘i:

$$R\left(x\right)=u_n\left(x\right)-\lambda\int_a^b K(x,t)u_n\left(t\right)dt-f\left(x\right)$$

tanlangan bazis funksiyalarga ortogonal bo‘lishi talab qilinadi:

$$\int_a^b R\left(x\right)\varphi_i\left(x\right)dx=0, i=1,2,\ldots,n$$

Bu usul sonli barqarorlik va yaqinlashuvni ta’minlab, zamonaviy hisoblash sxemalarining, jumladan chekli elementlar usulining nazariy asosini tashkil etadi.

Yaqinlashuv va xatolik tahlili

Sonli tahlilda asosiy masalalardan biri taxminiy yechimning haqiqiy yechimga yaqinlashuvini baholashdir:

$$\left\|u-u_n\right\|L_2\le C\inf_{v\in V_n}\left\|u-v\right\|L_2$$

Agar yechim va yadro yetarlicha silliq bo‘lsa, u holda yaqinlashuv tezligi yuqori bo‘ladi:

$$\left\|u-u_n\right\|=O\left(h^p\right)$$

Bu natijalar variatsion usullarning yuqori aniqlikka ega ekanligini ko‘rsatadi.

Misol:

$$u(x)=\sin(ax)+\frac{a}{2}\int_0^{\frac{\pi}{2a}}\cos(ax)u(t)dt$$

$$K(x,t)=\frac{a}{2}\cos(ax)$$

$$u=f+Ku,\ f(x)=\sin(ax)$$

$$J(u)=\frac{1}{2}\| u-Ku-f \|_{L_2}^2$$

$$u_N(x)=\sum_{K=1}^{N} c_K\varphi_K(x)$$

$$\varphi_1(x)=1,\quad \varphi_2(x)=\sin(ax)$$

Bu yerda n=2 deb olsak:

$$\frac{\partial J}{\partial c_i}=0$$

Galyokin shartiga teng:

$$\langle\, u-Ku-f,\varphi_i\rangle\ =0$$

Ya’ni:

$$\int_0^{\frac{\pi}{2a}}\left(u_N(x)-\frac{a}{2}\cos(ax)\int_0^{\frac{\pi}{2a}} u_N(t)dt-\sin(ax)\right)\cdot\varphi_i(x)dx=0$$

Integral hadlarini hisoblaymiz: $u_N(t)$ ni integralini hisoblaymiz:

$$\int_0^{\frac{\pi}{2a}} u_N(t)dt=c_1\frac{\pi}{2a}+c_2\int_0^{\frac{\pi}{2a}}\sin(at)dt$$

$$\int_0^{\frac{\pi}{2a}}\sin(at)dt=\frac{1}{a}$$

$$I=c_1\frac{\pi}{2a}+\frac{c_2}{a}$$

$$Ku_N(x)=\frac{a}{2}\cos(ax)\left(\ c_1\frac{\pi}{2a}+\frac{c_2}{a}\right)$$

$$\begin{cases} \langle u_N, 1\rangle - \langle Ku_N, 1\rangle = \langle f, 1\rangle \\ \langle u_N, \sin(ax)\rangle - \langle Ku_N, \sin(ax)\rangle = \langle f, \sin(ax)\rangle \end{cases}$$

Bundan quyidagilar kelib chiqadi:

$$\begin{cases} c_1 \dfrac{\pi}{2a} + c_2 \dfrac{1}{a} - \dfrac{\pi}{4} c_1 = \dfrac{1}{a} \\ c_2 \dfrac{\pi}{4a} = \dfrac{\pi}{4a} \end{cases}$$

$$c_2 = 1.\ c_1\left(\frac{\pi}{2a} - \frac{\pi}{4}\right) = \frac{1}{a} - 1\cdot\ \frac{1}{a} \Rightarrow c_1 = 0$$

Sonli (variatsion) yechim:

$$u(x) \approx \sin(ax)$$

Mazkur maqolada variatsion prinsiplar asosida integral tenglamalarni yechishning sonli tahlili ko‘rib chiqildi. Integral tenglamalarni variatsion qo‘yish ularni funksional analiz apparati yordamida o‘rganish imkonini berishi ko‘rsatildi. Rits va Galerkin usullari integral tenglamalarni yechishda barqaror, yaqinlashuvchi va hisoblash jihatdan samarali usullar ekanligi asoslandi.

Variatsion yondashuv nafaqat sonli yechim olish, balki yechimning nazariy xossalarini chuqur tahlil qilish imkonini beradi. Shu sababli bu usullar zamonaviy hisoblash matematikasi va muhandislik masalalarida muhim ahamiyatga ega.

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