A DIRICHLET-TYPE PROBLEM FOR AN ELLIPTIC EQUATION WITH A SINGULAR COEFFICIENT IN THE FIRST QUADRANT

Karimov K.T., Mirzaliyev M.M.

Fergana State University karimovk80@gmail.com, mirzaliyevgrand2020@gmail.com

Abstract. In this work, a Dirichlet-type problem for an elliptic equation with one singular coefficient in the first quadrant is investigated. The unique solvability of the posed problem is proven in the class of regular solutions. The uniqueness of the solution to the posed problem is proven by the energy integrals method, and the existence of the solution is proven by the Hankel integral transform method.

I. Introduction. Problem Formulation

In the domain $\Omega=\{(x,y):x\in(0,+\infty),y\in(0,+\infty)\}$, we consider an equation of elliptic type with one singular coefficient:

$$L_\beta(u)=u_{xx}+u_{yy}+\frac{2\beta}{y}u_y=0, \tag{1}$$

where $u=u(x,y)$ is the unknown function, and $\beta\in\mathbb{R}$, with $0<\beta<1/2$. In the domain Ω, equation (1) belongs to the elliptic type. The line $y=0$ is the line of singularity for the coefficient of the equation.

In this work, the unique solvability of the following problem is investigated:

Problem D_∞. Find a solution to equation (1) in the domain Ω satisfying the conditions:

$$u(x,y)\in C(\overline{\Omega})\cap C^{2,2}_{x,y}(\Omega), \quad y^{2\beta}u_y\in C(\overline{\Omega}), \tag{2}$$

$$u(0,y)=0, \quad \lim_{x\to+\infty}u(x,y)=0, \quad y\in[0,+\infty), \tag{3}$$

$$u(x,0)=\tau(x), \quad \lim_{y\to+\infty}u(x,y)=0, \quad x\in[0,+\infty), \tag{4}$$

where $\overline{\Omega}=\{(x,y):x\in[0,+\infty),y\in[0,+\infty)\}$, and $\tau(y)$ is a given function.

Investigations of stationary processes of various physical natures usually lead to problems for equations of elliptic type. The theory of such problems has a rich history and is one of the rapidly developing parts of the theory of partial differential equations.

Below, we provide historical data on boundary value problems for multidimensional elliptic equations with singular coefficients. For instance, in 1937, C. Agostinelli [1] considered the Dirichlet problem for a three-dimensional elliptic equation with one singular coefficient in a half-space. M.N. Olevsky [2] in 1949 announced an article in which the solution of the Dirichlet problem is written in explicit form in a multidimensional hemisphere for the equation $\Delta u + (p / x_n)u_{x_n} = f$, where f is a given function and Δ is the n-dimensional Laplace operator.This problem for the equation $\Delta u + (a / x_n)u_{x_n} - b^2 u = 0$, $a < 1$ in the half-space $x_n > 0$ was solved by M.B. Kapilevich [3] in 1952. After these works, interest in studying boundary value problems for elliptic equations with singular coefficients on the plane increased. Omitting a huge number of publications, we note some works most relevant to the current study. In the method of spectral analysis and Green's functions, boundary value problems for the Helmholtz equation and degenerate differential equations were investigated in works [4]–[9] and others.

In the present work, the posed problem D_∞ is investigated using the Hankel transform method. Many problems of physics, applied mathematics, and mathematical modeling reduce to solving differential, integral, and integro-differential equations. One of the effective methods for finding analytical solutions is the method of integral transforms. At the same time, the most studied and frequently applied of all Bessel transforms is the Hankel integral transform.

The Hankel integral transform of order ν for a function $f(t)$ is called the integral [10, 11]:

$$\bar{f}(p) = \int_0^{+\infty} f(t) t J_\nu(pt)dt, \quad \nu \geq -1/2, \quad 0 < p < +\infty. \tag{5}$$

Here $J_\nu(z)$ is the Bessel function of the first kind of order ν [12].

The Hankel transform of the function $f(t)$ is valid for any points on the interval $(0, +\infty)$ at which the function $f(t)$ is continuous or piecewise continuous with a finite number of jump discontinuities, and $\int_0^{+\infty} |f(t)| t^{1/2} dt < +\infty$.

The inversion formula for the Hankel transform is defined by the integral:

$$f(t)=\int_0^{+\infty}\overline{f}(p)pJ_\nu(pt)dp, \quad 0<t<+\infty. \tag{6}$$

The function $\overline{f}(p)$ is often called the Fourier-Bessel-Hankel transform [10], and the function $f(t)$ the original.

It is obviously expedient to apply the Hankel transform in the case when the variables in the equation change from 0 to $+\infty$. The posed problem D_∞ is a corresponding example.

II. Uniqueness of the solution

Theorem 1. If a solution to problem D_∞ exists, then it is unique.

Proof. Suppose that problem D_∞ has two solutions $u_1(x,y)$ and $u_2(x,y)$. Then $u(x,y)=u_1(x,y)-u_2(x,y)$ satisfies equation (1) and homogeneous boundary conditions. Let us prove that $u(x,y)\equiv 0$ in $\overline{\Omega}$. In the domain Ω, the identity is valid:

$$y^{2\beta}uL_\beta(u)=(y^{2\beta}uu_x)_x+(y^{2\beta}uu_y)_y-y^{2\beta}(u_x^2+u_y^2)=0.$$

Integrating this identity over the domain $\Omega_{\delta_1\delta_3}^{\delta_2\delta_4}=\{(x,y,z):\delta_1<x<\delta_2,\delta_3<y<\delta_4\}$, where $\delta_j, j=\overline{1,4}$ are sufficiently small numbers, we have:

$$\iint\limits_{\Omega_{\delta_1\delta_3}^{\delta_2\delta_4}}\left[(y^{2\beta}uu_x)_x+(y^{2\beta}uu_y)_y\right]dxdy=\iint\limits_{\Omega_{\delta_1\delta_3}^{\delta_2\delta_4}}y^{2\beta}(u_x^2+u_y^2)dxdy. \tag{7}$$

Obviously, if $\delta_1,\delta_3\to 0$, $\delta_2,\delta_4\to\infty$, then $\Omega_{\delta_1\delta_3}^{\delta_2\delta_4}\to\Omega$.

Applying the Gauss-Ostrogradsky formula to the left side of equality (7), we have:

$$\int\limits_{\delta_3}^{\delta_4}y^{2\beta}\left[u\;\delta_2,y\;u_x\;\delta_2,y\;-u\;\delta_1,y\;u_x\;\delta_1,y\;\right]dy+$$

$$+\int\limits_{\delta_1}^{\delta_2}\left[\delta_4^{2\beta}u\;x,\delta_4\;u_y\;x,\delta_4\;-\delta_3^{2\beta}u\;x,\delta_3\;u_y\;x,\delta_3\;\right]dx=$$

$$=\iiint\limits_{\Omega_{\delta_1\delta_3}^{\delta_2\delta_4}}\left[x^{2\alpha}y^{2\beta}\;u_x^2+u_y^2\;\right]dxdy.$$

Passing to the limit as $\delta_1, \delta_3 \to 0, \delta_2, \delta_4 \to +\infty$ and taking into account conditions (2)-(4) (at $\tau(y) = 0$), from the last equality we obtain:

$$\iiint_{\Omega} \left[x^{2\alpha} y^{2\beta} (u_x^2 + u_y^2) \right] dxdy = 0.$$

Consequently,

$$u_x(x, y) \equiv u_y(x, y) \equiv 0, (x, y) \in \Omega .$$

Then $u(x, y) \equiv const$, $(x, y) \in \Omega$. Since $u(x, y) \in C(\bar{\Omega})$ and $\lim\limits_{x\to\infty} x^{2\alpha} u(x, y) = 0$, then $u(x, y) \equiv 0, (x, y) \in \bar{\Omega}$ in $\bar{\Omega}$. From this follows the statement of the theorem.

III. Existence of the solution to problem D_∞. For the investigation of the question of existence of the solution to the original equation (1), we use the method of integral transforms. Let us apply the Hankel transform with respect to the variables x and y, defining the image of function $u(x, y)$ by the following relation:

$$\tilde{u}_\lambda^{\varepsilon_1 \varepsilon_2} = \int_{\varepsilon_1}^{\varepsilon_2} y^{1/2+\beta} u(x, y) J_{1/2-\beta}(\lambda y) dy . \tag{8}$$

Doubled differention of expression (8)with respect to variable x allows introducing the differentiation operator under the integral sign:

$$(\tilde{u}_\lambda^{\varepsilon_1 \varepsilon_2})'' = \int_{\varepsilon_1}^{\varepsilon_2} y^{1/2+\beta} u_{xx}(x, y) J_{1/2-\beta}(\lambda y) dy \tag{9}$$

Using equation (1) for substituting the second derivative u_{xx}, we pass to the expression containing derivatives with respect to y:

$$(\tilde{u}_\lambda^{\varepsilon_1 \varepsilon_2})'' = -\int_{\varepsilon_1}^{\varepsilon_2} \left(u_{yy} + \frac{2\beta}{y} u_y \right) y^{1/2+\beta} J_{1/2-\beta}(\lambda y) dy =$$

$$= -\int_{\varepsilon_1}^{\varepsilon_2} u_{yy} y^{1/2+\beta} J_{1/2-\beta}(\lambda y) dy - 2\beta \int_{\varepsilon_1}^{\varepsilon_2} u_y y^{\beta-1/2} J_{1/2-\beta}(\lambda y) dy, \tag{10}$$

where $\varepsilon_i, i = \overline{1,4}$ are positive numbers.

Obviously, $\lim\limits_{\substack{\varepsilon_1, \varepsilon_3 \to 0 \\ \varepsilon_2, \varepsilon_4 \to +\infty}} \tilde{u}_{\lambda \varepsilon_1 \varepsilon_3}^{\varepsilon_2 \varepsilon_4} = \tilde{u}_\lambda(x)$. Substituting into the obtained equation the expression for u_{xx} from the original differential equation, we pass to the consideration of the integral sum. Let us represent the obtained expression in the form of a sum of two integrals I_1 and I_2:

$$-\int_{\varepsilon_1}^{\varepsilon_2}\left(u_{yy}+\frac{2\beta}{y}u_y\right)y^{1/2+\beta}J_{1/2-\beta}(\lambda y)dy=$$

$$=\underbrace{-\int_{\varepsilon_1}^{\varepsilon_2}u_{yy}y^{1/2+\beta}J_{1/2-\beta}(\lambda y)dy}_{I_1}\underbrace{-2\beta\int_{\varepsilon_1}^{\varepsilon_2}yu_y y^{\beta-1/2}J_{1/2-\beta}(\lambda y)dy}_{I_2}.$$

For the calculation of integral I_1, we apply the integration by parts procedure. Using the recurrence relations for the derivatives of Bessel functions of the form $\frac{d}{dx}[x^{\nu}J_{\nu}(x)]=x^{\nu}J_{\nu-1}(x)$ and the rule for differentiating a product of functions, we can lower the order of the derivative of the function $u(x,y)$. Successive application of this method allows isolating the non-integral terms and reducing the problem to an integral from the function $u(x,y)$ itself.

$$I_1=\int_{\varepsilon_1}^{\varepsilon_2}y^{1/2+\beta}J_{1/2-\beta}(\lambda y)du_y=-u_y y^{1/2+\beta}J_{1/2-\beta}(\lambda y)\Big|_{\varepsilon_1}^{\varepsilon_2}+\int_{\varepsilon_1}^{\varepsilon_2}u_y\ \ y^{1/2+\beta}J_{1/2-\beta}(\lambda y)\ '\,dy=$$

$$=-u_y y^{1/2+\beta}J_{1/2-\beta}(\lambda y)\Big|_{\varepsilon_1}^{\varepsilon_2}+\int_{\varepsilon_1}^{\varepsilon_2}u_y\ \ y^{2\beta}\cdot y^{1/2-\beta}J_{1/2-\beta}(\lambda y)\,dy-$$

$$-\int_{\varepsilon_1}^{\varepsilon_2}y^{1/2+\beta}J_{1/2-\beta}(\lambda y)\Big|_{\varepsilon_1}^{\varepsilon_2}+\int_{\varepsilon_1}^{\varepsilon_2}u_y\ \ 2\beta y^{2\beta-1}y^{1/2-\beta}J_{1/2-\beta}(\lambda y)+y^{2\beta}\ \ \lambda y^{1/2-\beta}J_{-1/2-\beta}(\lambda y)\ \ dy.$$

After grouping the terms and reuse of formulas for differentiation of Bessel functions, in particular $\frac{d}{dx}[x^{-\nu}J_{\nu}(x)]=-x^{-\nu}J_{\nu+1}(x)$, we obtain a compact expression for the image of the function.

The final stage is the implementation of the limit transition at $\varepsilon_1\to 0$ and $\varepsilon_2\to\infty$. This allows extending the solution to the entire semi-infinite domain. Taking into account the asymptotic behavior of Bessel functions and the boundary conditions of the problem, we arrive at the form:

$$I_1 = -u_y y^{1/2+\beta} J_{1/2-\beta}(\lambda y)\Big|_{\varepsilon_1}^{\varepsilon_2} + 2\beta\int_{\varepsilon_1}^{\varepsilon_2} u_y y^{\beta-1/2} J_{1/2-\beta}(\lambda y)dy + \lambda\int_{\varepsilon_1}^{\varepsilon_2} u_y y^{\beta+1/2} J_{-1/2-\beta}(\lambda y)dy =$$

$$= -u_y y^{1/2+\beta} J_{1/2-\beta}(\lambda y)\Big|_{\varepsilon_1}^{\varepsilon_2} + I_2 + \lambda u_y y^{1/2+\beta} J_{-1/2-\beta}(\lambda y)\Big|_{\varepsilon_1}^{\varepsilon_2} + \lambda^2\int_{\varepsilon_1}^{\varepsilon_2} u y^{1/2+\beta} J'_{1/2-\beta}(\lambda y)dy.$$

Now, once more using the differentiation formula for Bessel functions $\frac{d}{dx}[x^{-\nu}J_\nu(x)] = -x^{-\nu}J_{\nu+1}(x)$ in the last integral, we obtain the following:

$$\tilde{u}_\lambda^{\varepsilon_1,\varepsilon_2} = \left[-u_y J^{(\lambda y)}_{\frac{1}{2}-\beta} + \lambda u_y J^{(\lambda y)}_{-\frac{1}{2}-\beta}\right] y^{\beta+\frac{1}{2}}\Bigg|_{\varepsilon_1}^{\varepsilon_2} + I_2 + \lambda^2\int_{\varepsilon_1}^{\varepsilon_2} u y^{\frac{1}{2}+\beta} J^{(\lambda y)'}_{\frac{1}{2}-\beta} dy - I_2 =$$

$$\langle \varepsilon_1 \to 0, \varepsilon_2 \to \infty \rangle$$

$$\tilde{u}_\lambda''(x) = -\lim_{\varepsilon_2\to\infty} \varepsilon_2^{\beta+1/2} J_{1/2-\beta}\ \lambda\varepsilon_2\ u_y\ x,\varepsilon_2\ + \lambda \lim_{\varepsilon_2\to\infty} \varepsilon_2^{\beta+\frac{1}{2}} J_{-1/2-\beta}\ \lambda\varepsilon_2\ u\ x,\varepsilon_2\ +$$

$$+ \lim_{\varepsilon_1\to 0} \varepsilon_1^{\beta+1/2} J_{1/2-\beta}\ \lambda\varepsilon_1\ u_y\ x,\varepsilon_1\ +$$

$$+ \lim_{\varepsilon_1\to 0} \varepsilon_1^{\beta+1/2} J_{-1/2-\beta}\ \lambda\varepsilon_1\ u\ x,\varepsilon_1\ + \lambda^2\int_0^\infty y^{\beta+1/2} u J_{1/2-\beta}\ \lambda y\ dy.$$

After completion of the limit transition and taking into account the conditions on the boundaries of the domain, the problem for function $\tilde{u}_\lambda(x)$ is reduced to solving a second-order homogeneous ordinary differential equation:

$$\tilde{u}_\lambda''(x) - \lambda^2 u_\lambda(x) = 0, \quad x \in (0,\infty)$$

For the unique determination of the solution, we formulate the corresponding boundary conditions for the image. On the basis of the given conditions for function $u(x,y)$ at $x \to 0$ and in the assumption of decay of the solution at infinity, we have:

$$\tilde{u}_\lambda(0) = \tau_{2\lambda}, \quad \tilde{u}_\lambda(\infty) = 0$$

where $\tau_{2\lambda} = \int_0^\infty y^{\frac{1+\beta}{2}} \tau_2(y) J_{\frac{1-\beta}{2}}(\lambda y)dy$.

IV. Construction of the general solution and inverse transform. The general solution of the obtained ODE has the form:

$$\tilde{u}_\lambda(x) = C_1 e^{\lambda x} + C_2 e^{-\lambda x}$$

From the condition of boundedness of the solution at infinity, it follows that the coefficient $C_1 = 0$. Satisfying the condition at $x = 0$, we finally obtain:

$$\tilde{u}_\lambda(x) = \tau_{2\lambda} e^{-\lambda x}$$

Let us apply the formula for the inverse Hankel transform, we obtain:

$$u(x,y) = \int_0^\infty \lambda \tilde{u}_\lambda(x) y^{1/2-\beta} J_{1/2-\beta}(\lambda y) d\lambda$$

Substituting the found expression for $\tilde{u}_\lambda(x)$ and expanding the value of $\tau_{2\lambda}$, we obtain the final integral representation of the Dirichlet problem solution in the form of an integral:

$$u(x,y) = y^{\frac{1-\beta}{2}} \int_0^\infty e^{-\lambda x} \tau_{2\lambda} \lambda J_{\frac{1}{2}-\beta}(\lambda y) d\lambda$$

This formula allows calculating the values of the sought function at any point of the considered domain through the given boundary data.

Conclusion

In this work, a Dirichlet-type problem for an elliptic equation with one singular coefficient in the first quadrant of the plane was investigated. During the study, the following scientific results were obtained:

Uniqueness of the solution: By the method of energy integrals, it is proven that if a regular solution to the posed problem exists, then it is unique.

Existence of the solution: With the help of the apparatus of the Hankel integral transform, the solution to the problem was found in explicit form and represented in the form of a double integral through the given boundary function.

Methodology: The effectiveness of using the double Hankel transform for reducing partial differential equations with singular coefficients to an ordinary differential equation was demonstrated.

The obtained results confirm the correctness of the formulation of the Dirichlet problem in unbounded domains and can be used for further study of the properties of solutions of this class of equations, as well as for the construction of approximate numerical methods.

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