IKKI O‘LCHOVLI ISSIQLIK TENGLAMASINI AYIRMALI SXEMA YORDAMIDA SONLI YECHISH VA DASTURIY REALIZATSIYASI

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Annotatsiya. Mazkur maqolada ikki o‘lchovli issiqlik tenglamasini sonli usullar yordamida yechish masalasi ko‘rib chiqilgan. Issiqlik tarqalish jarayonlarini modellashtirishda ayirmali sxemalardan foydalanish samarali hisoblanadi. Tadqiqotda aniq (explicit) ayirmali sxema asosida tenglama diskretlashtirildi va Python hamda MATLAB muhitlarida dasturiy realizatsiya qilindi. Olingan natijalar grafik ko‘rinishda tahlil qilinib, usulning samaradorligi ko‘rsatildi.

Kalit so‘zlar: issiqlik tenglamasi, sonli usullar, ayirmali sxema, explicit sxema, MATLAB, Python, modellashtirish.

Kirish. Issiqlik tarqalish jarayonlarini modellashtirish zamonaviy ilm-fan va texnologiyaning muhim yo‘nalishlaridan biri hisoblanadi. Bunday jarayonlar energetika, qurilish, materialshunoslik, mikroelektronika va boshqa ko‘plab sohalarda keng qo‘llaniladi. Issiqlik almashinuvi jarayonlarini matematik tavsiflash uchun eng muhim modellardan biri – bu issiqlik o‘tkazuvchanlik tenglamasi bo‘lib, u parabolik tipdagi xususiy hosilali differensial tenglamalar sinfiga kiradi.

Ko‘p hollarda real fizik jarayonlar murakkab geometrik sohalarda va noaniq boshlang‘ich-chegaraviy shartlar ostida kechadi. Shu sababli issiqlik tenglamasining analitik yechimini topish amaliy jihatdan qiyin yoki imkonsiz bo‘ladi. Aynan shu holatda sonli usullar, xususan, ayirmali sxemalar muhim ahamiyat kasb etadi.

So‘nggi yillarda hisoblash texnikasining rivojlanishi natijasida xususiy hosilali differensial tenglamalarni sonli yechish usullari keng rivojlandi. Ular orasida ayirmali sxemalar soddaligi, hisoblash samaradorligi va dasturiy realizatsiya qilish qulayligi bilan ajralib turadi. Ayniqsa, ikki o‘lchovli issiqlik tenglamasini yechishda aniq (explicit) sxemalar tezkor hisoblash imkonini beradi.

Shu bilan birga, aniq sxemalar uchun barqarorlik sharti muhim rol o‘ynaydi. Agar bu shart bajarilmasa, sonli yechim noto‘g‘ri natijalar berishi yoki divergensiyaga uchrashi mumkin. Shuning uchun sxemani qurishda uning barqarorligi, yaqinlashuvi va xatolik tahlili alohida e’tibor talab qiladi.

Mazkur ishning asosiy maqsadi – ikki o‘lchovli issiqlik tenglamasini aniq ayirmali sxema yordamida diskretlashtirish, uning barqarorlik va yaqinlashuv xossalarini tahlil qilish hamda Python va MATLAB muhitlarida dasturiy realizatsiyasini amalga oshirishdan iborat.

Tadqiqot vazifalari quyidagilardan iborat:

- ikki o‘lchovli issiqlik tenglamasining matematik modelini shakllantirish;
- ayirmali sxema asosida diskret modelni qurish;
- sxemaning barqarorlik va yaqinlashuv shartlarini aniqlash;
- algoritm ishlab chiqish va uni dasturiy realizatsiya qilish;
- olingan natijalarni grafik ko‘rinishda tahlil qilish.

Mazkur tadqiqotning ilmiy yangiligi shundan iboratki, aniq ayirmali sxema asosida qurilgan model real hisoblash muhitlarida (Python va MATLAB) implementatsiya qilinib, natijalar solishtirma tahlil qilinadi hamda sxemaning samaradorligi amaliy jihatdan asoslab beriladi.

Amaliy misol.

Ikki o‘lchovli issiqlik tenglamasining amaliy qo‘llanishini ko‘rsatish maqsadida kvadrat shakldagi yupqa metall plastinada issiqlik tarqalish jarayoni qaraladi. Plastina quyidagi sohada aniqlangan:

$$0 \le x \le 1, \qquad 0 \le y \le 1, \qquad t > 0.$$

Plastinadagi temperatura taqsimoti

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x, y, t).$$

bu yerda $f\ x, y, t\ = x + y + t$ – plastina ichida mavjud bo‘lgan issiqlik manbai funksiyasidir.

Boshlang‘ich shart plastinadagi temperaturaning dastlabki taqsimotini belgilaydi:

$$u(x, y, 0) = \sin(\pi x)\cos(\pi y).$$

Bu shart plastinaning ichki nuqtalarida sinusoidal xarakterga ega temperatura taqsimoti mavjudligini ko‘rsatadi.

Chegaraviy shartlar plastinaning chekka qismlarida temperatura qanday boshqarilishini ifodalaydi:

$$u(x,0,t)=0,\ u(x,1,t)=xt,\ u(0,y,t)=yt,\ u(1,y,t)=yt.$$

Ushbu shartlarning fizik talqini quyidagicha:

- pastki chegara $y=0$ sovitilib turiladi va temperatura nolga teng saqlanadi;
- yuqori chegara $y=1$ bo‘ylab temperatura vaqt o‘tishi bilan ortadi va x bo‘yicha chiziqli taqsimlangan;
- chap va o‘ng chegaralarda temperatura y koordinata va vaqtga proporsional ravishda ortib boradi.

Shunday qilib, plastinaning turli qismlarida issiqlik oqimi notekis va vaqtga bog‘liq holda ta’sir qiladi.

Mazkur masalani sonli yechish uchun fazoviy soha bir xil qadamli to‘rga ajratiladi:

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To’r quramiz:

$$x_i = ih, y_j = jh, h = \frac{1}{N}, t^n = n\tau ,$$

bu yerda:

$$N = 20, h = \frac{1}{20}, \tau = \frac{h^2}{4}.$$

Bu ayirmali sxemaning barqarorlik shartidan kelib chiqadi: $\tau \le \frac{h^2}{4}$.

Fazoviy ikkinchi tartibli hosilalar markaziy ayirmalar orqali approksimatsiya qilinadi:

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{U_{i+1,j}^n - 2U_{i,j}^n + U_{i-1,j}^n}{h^2}$$

$$\frac{\partial^2 u}{\partial y^2} \approx \frac{U_{i,j+1}^n - 2U_{i,j}^n + U_{i,j-1}^n}{h^2}$$

$$\frac{\partial u}{\partial t} \approx \frac{U_{i,j}^{n+1} - U_{i,j}^n}{\tau}$$

$$U_{i,j}^{n+1} = U_{i,j}^n + \tau\left(\frac{U_{i+1,j}^n - 2U_{i,j}^n + U_{i-1,j}^n}{h^2} + \frac{U_{i,j+1}^n - 2U_{i,j}^n + U_{i,j-1}^n}{h^2} + f(x_i, y_j, t^n)\right)$$

Umumiy yaqinlashish tartibi: $O\left(\tau + h^2\right)$

Dasturiy realizatsiya: Masala Python va MATLAB muhitlarida realizatsiya qilindi. Hisoblash jarayonida 21×21 o‘lchamdagi to‘r ishlatildi va vaqt bo‘yicha iteratsiyalar bajarildi. Dasturda boshlang‘ich hamda chegaraviy shartlar hisobga olindi.

```
import numpy as np
N = 20
h = 1/N
tau = h*h/4
M = 100
x = np.linspace(0,1,N+1)
y = np.linspace(0,1,N+1)
U = np.zeros((N+1,N+1))
for i in range(N+1):
    for j in range(N+1):
        U[i,j] = np.sin(np.pi*x[i])*np.cos(np.pi*y[j])
for n in range(M):
    t = n*tau
    U_new = U.copy()

    for j in range(N+1):
        U_new[0,j] = y[j]*t
        U_new[N,j] = y[j]*t
    for i in range(N+1):
        U_new[i,N] = x[i]*t
        U_new[i,0] = 0
    for i in range(1,N):
        for j in range(1,N):
```

```
        U_new[i,j] = U[i,j] + tau*((U[i+1,j]-2*U[i,j]+U[i-
1,j])/h**2+(U[i,j+1]-2*U[i,j]+U[i,j-1])/h**2+x[i]+y[j]+t)
    U = U_new
```

MatLab :

```
clc; clear; close all;
%% Parametrlar
N = 20;
h = 1/N;
tau = h^2/4;
M = 100;
T = M*tau;
x = linspace(0,1,N+1);
y = linspace(0,1,N+1);
%% Boshlang'ich shart
U = zeros(N+1,N+1);   % 0..N
for i = 1:N+1
    for j = 1:N+1
        U(i,j) = sin(pi*x(i))*cos(pi*y(j));
    end
end
%% Vaqt bo'yicha hisoblash (explicit)
for n = 1:M
    t = (n-1)*tau;
    Unew = U;
    % Chegaraviy shartlar
    Unew(1,:)   = y*t;       % x=0
    Unew(end,:) = y*t;       % x=1
    Unew(:,end) = x'*t;      % y=1
    Unew(:,1)   = 0;         % y=0
    % Ichki nuqtalar
    for i = 2:N
        for j = 2:N
            Unew(i,j) = U(i,j) + tau*( ...
                (U(i+1,j)-2*U(i,j)+U(i-1,j))/h^2 + ...
                (U(i,j+1)-2*U(i,j)+U(i,j-1))/h^2 + ...
                x(i) + y(j) + t ...
            );
        end
```

```
    end
    U = Unew;
end
%%  Jadval ko'rinishida chiqarish
disp('21x21 sonli yechim jadvali:');
disp(U);
%%  3D grafik
[X,Y] = meshgrid(x,y);
figure;
surf(X,Y,U');
xlabel('x'); ylabel('y'); zlabel('U(x,y,T)');
title('2D issiqlik tenglamasining sonli yechimi');
shading interp;
colorbar;
```

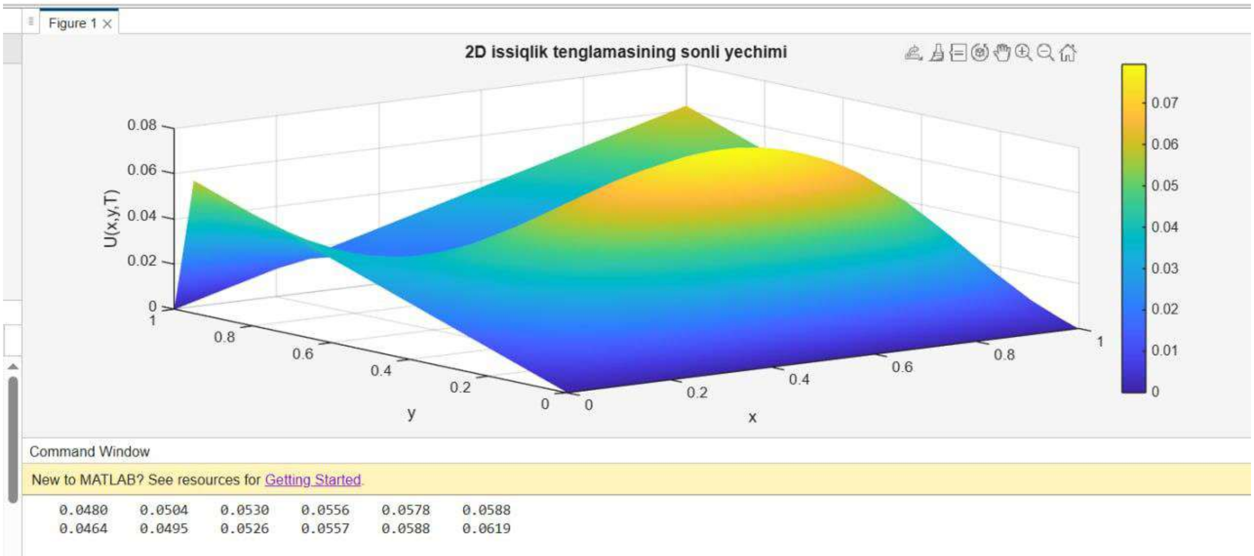
21x21 sonli yechim jadvali:

```
        0    0.0031    0.0062    0.0093    0.0124    0.0155    0.0186    0.0217    0.0248    0.0278
0.0309    0.0340    0.0371    0.0402    0.0433
        0    0.0058    0.0112    0.0161    0.0204    0.0241    0.0271    0.0294    0.0313    0.0327
0.0338    0.0348    0.0358    0.0370    0.0384
        0    0.0083    0.0160    0.0226    0.0282    0.0324    0.0353    0.0370    0.0376    0.0373
0.0365    0.0355    0.0346    0.0340    0.0338
        0    0.0107    0.0205    0.0289    0.0356    0.0404    0.0432    0.0442    0.0436    0.0418
0.0392    0.0363    0.0336    0.0313    0.0297
        0    0.0130    0.0248    0.0347    0.0425    0.0478    0.0506    0.0510    0.0493    0.0461
0.0419    0.0372    0.0329    0.0292    0.0264
        0    0.0151    0.0286    0.0400    0.0488    0.0546    0.0574    0.0572    0.0546    0.0501
0.0445    0.0384    0.0325    0.0276    0.0240
        0    0.0169    0.0321    0.0448    0.0544    0.0607    0.0634    0.0629    0.0595    0.0539
0.0470    0.0397    0.0327    0.0268    0.0224
        0    0.0185    0.0350    0.0488    0.0592    0.0659    0.0686    0.0678    0.0638    0.0574
0.0496    0.0412    0.0333    0.0266    0.0217
        0    0.0198    0.0374    0.0521    0.0631    0.0701    0.0729    0.0719    0.0675    0.0605
0.0519    0.0429    0.0343    0.0271    0.0219
        0    0.0208    0.0392    0.0546    0.0661    0.0733    0.0762    0.0751    0.0704    0.0631
0.0542    0.0447    0.0357    0.0282    0.0228
```

0 0.0214 0.0404 0.0561 0.0679 0.0754 0.0784 0.0773 0.0726 0.0652
0.0562 0.0466 0.0375 0.0300 0.0246
0 0.0217 0.0409 0.0568 0.0687 0.0762 0.0794 0.0784 0.0739 0.0667
0.0578 0.0485 0.0396 0.0322 0.0270
0 0.0216 0.0407 0.0564 0.0683 0.0758 0.0791 0.0784 0.0742 0.0675
0.0591 0.0502 0.0418 0.0348 0.0299
0 0.0211 0.0397 0.0551 0.0666 0.0741 0.0775 0.0771 0.0735 0.0674
0.0597 0.0516 0.0440 0.0376 0.0332
0 0.0203 0.0380 0.0526 0.0637 0.0709 0.0744 0.0744 0.0715 0.0663
0.0597 0.0526 0.0459 0.0404 0.0367
0 0.0190 0.0354 0.0490 0.0593 0.0662 0.0698 0.0703 0.0681 0.0640
0.0586 0.0529 0.0474 0.0430 0.0401
0 0.0172 0.0320 0.0442 0.0535 0.0600 0.0635 0.0645 0.0633 0.0604
0.0565 0.0522 0.0482 0.0450 0.0431
0 0.0149 0.0276 0.0380 0.0461 0.0519 0.0555 0.0570 0.0568 0.0553
0.0529 0.0503 0.0479 0.0462 0.0453
0 0.0120 0.0220 0.0304 0.0370 0.0420 0.0454 0.0475 0.0483 0.0483
0.0477 0.0469 0.0463 0.0460 0.0465
0 0.0082 0.0151 0.0209 0.0258 0.0299 0.0332 0.0358 0.0377 0.0392
0.0405 0.0416 0.0428 0.0442 0.0459
0 0.0031 0.0062 0.0093 0.0124 0.0155 0.0186 0.0217 0.0248 0.0278
0.0309 0.0340 0.0371 0.0402 0.0433

0.0464	0.0495	0.0526	0.0557	0.0588	0
0.0399	0.0412	0.0416	0.0397	0.0311	0.0031
0.0339	0.0339	0.0330	0.0298	0.0218	0.0062
0.0288	0.0281	0.0270	0.0243	0.0188	0.0093
0.0247	0.0238	0.0230	0.0214	0.0182	0.0124
0.0218	0.0207	0.0205	0.0201	0.0187	0.0155
0.0198	0.0189	0.0192	0.0198	0.0200	0.0186
0.0189	0.0181	0.0189	0.0204	0.0217	0.0217
0.0190	0.0183	0.0195	0.0217	0.0239	0.0248

0.0199	0.0194	0.0208	0.0235	0.0263	0.0278
0.0217	0.0213	0.0230	0.0259	0.0290	0.0309
0.0242	0.0239	0.0257	0.0288	0.0320	0.0340
0.0274	0.0272	0.0291	0.0321	0.0352	0.0371
0.0310	0.0310	0.0329	0.0357	0.0386	0.0402
0.0349	0.0351	0.0369	0.0396	0.0421	0.0433
0.0389	0.0393	0.0411	0.0435	0.0457	0.0464
0.0425	0.0433	0.0452	0.0474	0.0492	0.0495
0.0456	0.0468	0.0488	0.0509	0.0525	0.0526
0.0476	0.0493	0.0516	0.0538	0.0555	0.0557
0.0480	0.0504	0.0530	0.0556	0.0578	0.0588
0.0464	0.0495	0.0526	0.0557	0.0588	0.0619



Grafik tahlili shuni ko‘rsatadiki - yechim silliq funksiya, chegaraviy shartlar bajarilgan, barqarorlik saqlangan va manba had $(x + y + t)$ tufayli sirt yuqoriga ko‘tarilgan.

Explicit sxemaning afzalliklari: algoritm sodda, dasturlash oson, kichik o‘lchamli masalalarda tez ishlaydi; kamchiligi esa barqarorlik sharti vaqt qadamini cheklaydi va katta T uchun juda ko‘p iteratsiya talab etiladi.

Xulosa. Ikki o‘lchovli issiqlik tenglamasi aniq ayirmali sxema yordamida muvaffaqiyatli sonli yechildi. Ikkita dasturiy muhitda olingan natijalar bir-biriga mos keladi. Explicit sxema kichik vaqt oralig'ida samarali ishlaydi, biroq katta vaqt qadamlarda implicit sxemalar ustunlikka ega. Mazkur tadqiqot natijalari issiqlik jarayonlarini modellashtirishda sonli usullarning samaradorligini ko'rsatadi.

Foydalanilgan adabiyotlar ro'yxati.

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