UCH O‘LCHOVLI UCHTA SINGULYAR KOEFFITSIENTGA EGA BO‘LGAN ELLIPTIK TIPDAGI TENGLAMA UCHUN SHARNING SAKKIZDAN BIR BO‘LAGIDA NEYMAN MASALASINING SPEKTRAL XOSSALARI

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Annotatsiya. Ushbu ishda uch o‘lchovli fazoda uchta singulyar koeffitsientga ega elliptik tipdagi differensial tenglama uchun Neyman masalasining xos qiymatlari va xos funksiyalari topilgan. Masala markazi koordinatalar boshida joylashgan sharning sakkizdan bir qismida qaraladi. O‘zgaruvchilarni ajratish usuli yordamida masala uchta oddiy differensial tenglamalar uchun xos qiymat haqidagi masalalarga keltiriladi va xos qiymatlar hamda ularga mos xos funksiyalar aniqlanadi. Xos funksiyalar sferik koordinatalarda ifodalanib, ular Gaussning gipergeometrik funksiyalari orqali aniq ko‘rinishda yoziladi. Olingan yechimlar parametrlarga bog‘liq holda berilib, ularning asosiy xossalari ko‘rsatib o‘tiladi. Natijalar singulyar koeffitsientli elliptik tenglamalar nazariyasida va chegaraviy masalalarni yechishda qo‘llanilishi mumkin.

Kalit so‘zlar: Neyman masalasi, uch o‘lchamli elliptik tenglama, singulyar koeffitsientlar, xos funksiyalar, gipergeometrik funksiyalar, sferik koordinatalar, spektral masala.

KIRISH. Elliptik turdagi differensial tenglamalar nazariyasi matematik-fizika va muhandislik sohalarida keng qo‘llaniladi. Ayniqsa, uch o‘lchamli fazoda singulyar koeffitsientlarga ega elliptik tenglamalar chegaraviy masalalarni yechishda muhim ahamiyatga ega. Ushbu ishda markazi koordinatalar boshida joylashgan sharning sakkizdan bir qismida Neyman masalasi ko‘rib chiqiladi.

Masalani yechishda o‘zgaruvchilarni ajratish usuli qo‘llanilib, u spektral masalaga keltiriladi. Shuningdek, xos qiymatlar va xos funksiyalar aniqlanadi, ular sferik koordinatalarda ifodalanadi va Gauss gipergeometrik funksiyalari orqali aniq ko‘rinishda yoziladi. Ushbu yondashuv singulyar koeffitsientli elliptik tenglamalarni tahlil qilish va chegaraviy masalalarni yechishda samarali vosita sifatida xizmat qiladi.

Ushbu maqola tadqiqotning nazariy va amaliy jihatlarini birlashtiradi hamda olingan natijalar matematika va fizika sohasidagi muammolarni hal qilishda qo‘llanilishi mumkin.

ADABIYOTLAR TAHLILI VA METODOLOGIYA. [2], [3] ishlarda ikkinchi tartibli elliptik tipdagi tenglamalar uchun cheklangan sohada Neyman masalasining umumlashgan yechimlari batafsil o‘rganilgan. [4], [5], [6] ishlarda ko‘p o‘lchovli Laplas tenglamasi uchun silindrik sohada klassik yechimlar aniq ko‘rinishda berilgan. Shu bilan birga, ikkita singulyar koeffitsientga ega uch o‘lchovli elliptik tenglamalar uchun ham xuddi shunday ishlar [7], [8], [9] da ham o‘rganilgan.

NATIJA VA MUHOKAMA. Uch o‘lchovli Ω soha quyidagi

$$S=\{(x,y,z):x^2+y^2+z^2=a^2, 0<x<a,\ 0<y<a,\ 0<z<a\}$$

sfera qismi va

$$B_1=\{(x,y,z):y=0,\ x^2+z^2<a^2, 0<x<a, 0<z<a\}$$

$$B_2=\{(x,y,z):z=0,\ x^2+y^2<a^2, 0<x<a, 0<y<a\}$$

$$B_3=\{(x,y,z):x=0,\ x^2+y^2<a^2, 0<y<a, 0<z<a\}$$

chorak doiralardan tashkil topgan.

Ω sohada quyidagi

$$L^{\lambda}_{\alpha\beta\gamma}u\equiv u_{xx}+u_{yy}+u_{zz}+\frac{2\alpha}{x}u_x+\frac{2\beta}{y}u_y+\frac{2\gamma}{z}u_z+\lambda u=0 \qquad (1)$$

uch o‘lchovli uchta singulyar koeffitsientga ega bo‘lgan tenglamani qaraymiz, bu yerda $u=u(x,y,z)$ - noma’lum funksiya, $\alpha,\beta,\gamma\in R$, jumladan, $\alpha,\beta,\gamma\in\ 0,1/2\ $, λ-sonli parametr.

(1) tenglama Ω sohada elliptik tipga tegishli bo‘lib, bu sohaning chegaralari bo‘lmish B_1, B_2 va B_3 yarimdoiralar bu tenglama uchun singulyarlik (buzilish) tekisliklari hisoblanadi.

N_λ masala. λ parametrning shunday qiymatlari topilsinki, (1) tenglamaning $C\ \overline{\Omega}\ \cap C^{2,2,2}_{x,y,z}\ \Omega$ sinfga tegishli va

$$\lim_{x\to 0}x^{2\alpha}u_x\ \ x,y,z\ =0,\ \ x,y,z\ \in B_3 \qquad (2)$$

$$\lim_{y\to 0}y^{2\beta}u_y\ \ x,y,z\ =0,\ \ x,y,z\ \in B_1 \qquad (3)$$

$$\lim_{z\to 0}z^{2\gamma}u_z\ \ x,y,z\ =0,\ \ x,y,z\ \in B_2 \qquad (4)$$

$$\left.\frac{\partial u}{\partial \vec{n}}\right|_{r=a} = 0, \quad (x,y,z) \in S \tag{5}$$

shartni qanoatlantiruvchi trivial bo‘lmagan $u(x,y,z)$ yechimi mavjud bo‘lsin.

Odatda λ ning N_λ masalada topilishi talab etilayotgan qiymatlari xos qiymatlar, bu qiymatlarga mos trivial bo‘lmagan funksiyalar masalaning xos funksiyalari deyiladi.

Masalaning tadqiqoti. Buning uchun Ω sohada (x,y,z) Dekart koordinatalari sistemasi bilan (r,θ,φ) sferik koordinatalar sistemasini bog‘lab turuvchi

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta$$

formuladan foydalanib, (1) tenglamani sferik koordinatalar sistemasida tasvirlab olamiz. Bu yerda $r = \sqrt{x^2+y^2+z^2}$, $\theta - \vec{t}(x,y,z)$, vektor bilan z o‘qi orasidagi burchak; $\varphi - \vec{t}(x,y,0)$ vektor bilan x o‘qi orasidagi burchak.

$u(x,y,z) = V(r,\theta,\varphi)$ almashtirishdan foydalanamiz. (1) da ishtirok etgan birinchi va ikkinchi tartibli barcha hosilalarni hisoblab chiqamiz.

$$u_x = \frac{x}{r}V_r + \frac{xz}{r^2\sqrt{r^2-z^2}}V_\theta - \frac{y}{x^2+y^2}V_\varphi;$$

$$u_y = \frac{y}{r}V_r + \frac{zy}{r^2\sqrt{r^2-z^2}}V_\theta + \frac{x}{x^2+y^2}V_\varphi;$$

$$u_z = \frac{z}{r}V_r - \frac{r^2-z^2}{r^2\sqrt{x^2+y^2}}V_\theta;$$

$$u_{xx} = \frac{x^2}{r^2}V_{rr} + \frac{2x^2z}{r^3\sqrt{r^2-z^2}}V_{r\theta} - \frac{2xy}{r(x^2+y^2)}V_{r\varphi} + \frac{x^2z^2}{r^4(x^2+y^2)}V_{\theta\theta} -$$

$$-\frac{2xyz}{r^2(x^2+y^2)\sqrt{r^2-z^2}}V_{\theta\varphi} + \frac{y^2}{x^2+y^2}V_{\varphi\varphi} + \frac{r^2-x^2}{r^3}V_r +$$

$$+\frac{zr^2\sqrt{x^2+y^2} - 2x^2z\sqrt{x^2+y^2} - x^2zr^2(x^2+y^2)^{-1/2}}{r^4(x^2+y^2)}V_\theta + \frac{2xy}{(x^2+y^2)^2}V_\varphi;$$

$$u_{yy}=\frac{y^2}{r^2}V_{rr}+\frac{2y^2z}{r^3\sqrt{r^2-z^2}}V_{r\theta}+\frac{2xy}{r\left(x^2+y^2\right)}V_{r\varphi}+\frac{y^2z^2}{r^4\left(r^2-z^2\right)}V_{\theta\theta}+$$

$$+\frac{2xyz}{r^2\left(x^2+y^2\right)\sqrt{r^2-z^2}}V_{\theta\varphi}+\frac{x^2}{x^2+y^2}V_{\varphi\varphi}+\frac{r^2-y^2}{r^3}V_r+$$

$$+\frac{zr^2\sqrt{x^2+y^2}-2y^2z\sqrt{x^2+y^2}-y^2zr^2\left(x^2+y^2\right)^{-1/2}}{r^4\left(x^2+y^2\right)}V_\theta-\frac{2xy}{\left(x^2+y^2\right)^2}V_\varphi;$$

$$u_{zz}=\frac{z^2}{r^2}V_{rr}+\frac{2z^2\left(x^2+y^2\right)}{r^3\sqrt{x^2+y^2}}V_{r\theta}+\frac{\left(x^2+y^2\right)^2}{r^4\left(x^2+y^2\right)}V_{\theta\theta}+$$

$$+\frac{r^2-z^2}{r^3}V_r+\frac{2z\left(x^2+y^2\right)}{r^4\sqrt{x^2+y^2}}V_\theta\ .$$

topilgan hosilalarni (1) tenglamaga olib borib qo‘yamiz va $V\left(r,\theta,\varphi\right)$ ga nisbatan

$$V_{rr}+\frac{2\left(1+\alpha+\beta+\gamma\right)}{r}V_r+\frac{\left(1+2\alpha+2\beta\right)ctg\theta-2\gamma tg\theta}{r^2}V_\theta+$$

$$+\frac{1}{r^2}V_{\theta\theta}+\frac{4\left(\beta\cos^2\varphi-\alpha\sin^2\varphi\right)}{r^2\sin^2\theta\sin2\varphi}V_\varphi+\frac{1}{r^2\sin^2\theta}V_{\varphi\varphi}+\lambda V=0 \qquad (6)$$

ko‘rinishdagi tenglamani hosil qilamiz.

(6) tenglamada o‘zgaruvchilarni ajratamiz. Bu jarayonni ikkita bosqichda bajaramiz. Birinchi bo‘lib (6) tenglamani $V\left(r,\theta,\varphi\right)=R\left(r\right)Q\left(\theta,\varphi\right)$ ko‘rinishda o‘zgaruvchilarini ajratib olamiz va $R\left(r\right)Q\left(\theta,\varphi\right)$ ifodaga bo‘lib, natijada quyidagi ko‘rinishdagi tenglamani hosil qilamiz:

$$\frac{1}{R\left(r\right)}\left[R''\left(r\right)+\frac{2\left(1+\alpha+\beta+\gamma\right)}{r}R'\left(r\right)+\lambda R\left(r\right)\right]=$$

$$=-\frac{1}{Q}\left[\frac{1}{r^2}Q_{\theta\theta}+\frac{\left(1+2\alpha+2\beta\right)ctg\theta-2\gamma tg\theta}{r^2}Q_\theta\right]+$$
$$+\left(-\frac{1}{Q}\right)\left[\frac{4\left(\beta\cos^2\varphi-\alpha\sin^2\varphi\right)}{r^2\sin^2\theta\sin2\varphi}Q_\varphi+\frac{1}{r^2\sin^2\theta}Q_{\varphi\varphi}\right] \qquad (7)$$

(7) tenglamada chap tomondagi o‘zgaruvchilar, o‘ng tomonda joylashgan o‘zgaruvchiga bog‘liq bo‘lmagan holda o‘zgarganda ham tenglik o‘rinli bo‘lganligi uchun bu tenglikni qandaydir o‘zgarmas χ songa tenglashtirishimiz mumkin. Bu yerdan ikkita tenglamani hosil qilamiz:

$$-\Big[Q_{\theta\theta}(\theta,\varphi)+\left[(1+2\alpha+2\beta)ctg\theta-2\gamma tg\theta\right]Q_{\theta}(\theta,\varphi)+$$
$$+\frac{4(\beta\cos^2\varphi-\alpha\sin^2\varphi)}{\sin^2\theta\sin 2\varphi}Q_{\varphi}(\theta,\varphi)+\frac{1}{\sin^2\theta}Q_{\varphi\varphi}(\theta,\varphi)\Big]=\chi Q(\theta,\varphi), \quad (8)$$

$$r^2R''(r)+2(1+\alpha+\beta+\gamma)rR'(r)+(\lambda r^2-\chi)R(r)=0,\ 0<r<a \quad (9)$$

Endi esa (8) tenglamada o‘zgaruvchilarni $Q(\theta,\varphi)=T(\theta)\Phi(\varphi)$ formula yordamida ajratamiz.

$$\frac{\sin^2\theta\left(T''(\theta)+\left[(1+2\alpha+2\beta)ctg\theta-2\gamma tg\theta\right]T'(\theta)\right)}{T(\theta)}+\chi\sin^2\theta=$$
$$=\frac{-\left[\Phi''(\varphi)+\dfrac{4(\beta\cos^2\varphi-\alpha\sin^2\varphi)}{\sin 2\varphi}\Phi'(\varphi)\right]}{\Phi(\varphi)}. \quad (10)$$

Bu yerda ham xuddi yuqoridagi holat, har ikkala tomonni μ o‘zgarmas soniga tenglashtiramiz. Natijada bizda:

$$\Phi''(\varphi)+\frac{4(\beta\cos^2\varphi-\alpha\sin^2\varphi)}{\sin 2\varphi}\Phi'(\varphi)+\mu\Phi(\varphi)=0,\ 0<\varphi<\pi/2, \quad (11)$$

$$T''(\theta)+\left[(1+2\alpha+2\beta)ctg\theta-2\gamma tg\theta\right]T'(\theta)+(\chi-\mu/\sin^2\theta)T(\theta)=0, \quad (12)$$
$$0<\theta<\pi/2$$

tenglamalar hosil bo‘ladi.

{(2)-(5)} shartlardan bu oddiy differensial tenglamalar uchun chegaraviy shartlarni keltirib chiqaramiz.

$$r^2R''(r)+2(1+\alpha+\beta+\gamma)rR'(r)+(\lambda r^2-\chi)R(r)=0,\ \ 0<r<a, \quad (13)$$

$$|R(0)|=+\infty,\ R'(a)=0; \quad (14)$$

$$T''(\theta)+\left[(1+2\alpha+2\beta)ctg\theta-2\gamma tg\theta\right]T'(\theta)+\left(\chi-\frac{\mu}{\sin^2\theta}\right)T(\theta)=0,\ 0<\theta<\frac{\pi}{2},\quad(15)$$

$$|T(0)|<+\infty,\ \lim_{\theta\to\frac{\pi}{2}}(\cos\theta)^{2\gamma}T'(t)=0\quad(16)$$

$$\Phi''(\varphi)+\frac{4(\beta\cos^2\varphi-\alpha\sin^2\varphi)}{\sin 2\varphi}\Phi'(\varphi)+\mu\Phi(\varphi)=0,\ \ 0<\varphi<\frac{\pi}{2},\quad(17)$$

$$\lim_{\varphi\to 0}(\sin\varphi)^{2\beta}\varPhi'(\varphi)=0,\ \lim_{\varphi\to\frac{\pi}{2}}(\cos\varphi)^{2\alpha}\varPhi'(\varphi)=0\quad(18)$$

Bu masalalarning tadqiqotiga o‘tamiz. Birinchi bo‘lib, {(17), (18)} masalaning xos qiymatlari va ularga mos keluvchi trivial bo‘lmagan xos funksiyalarini topamiz. Buning uchun (17) tenglamada $z=\sin^2\varphi$ almashtirish bajaramiz va natijada Gausssning gipergeometrik tenglamasi deb ataluvchi tenglamani hosil qilamiz:

$$z(1-z)\tilde{\Phi}''(z)+\left[\frac{1}{2}+\beta-(1+\alpha+\beta)z\right]\tilde{\Phi}'(z)+\frac{\mu}{4}\tilde{\Phi}(z)=0,\quad(19)$$

bu yerda $\tilde{\Phi}(z)=\Phi(\arcsin\sqrt{z})$.

(19) tenglamaning umumiy yechimini topamiz va belgilashdan orqaga qaytamiz:

$$\Phi(\varphi)=AF\left[(\alpha+\beta+\omega)/2,\ (\alpha+\beta-\omega)/2,1/2+\beta;\sin^2\varphi\right]+$$

$$+B(\sin\varphi)^{1-2\beta}F\left[(1+\alpha-\beta+\omega)/2,\ (1+\alpha-\beta-\omega)/2,3/2-\beta;\sin^2\varphi\right],\quad(20)$$

bu yerda $\omega=\sqrt{(\alpha+\beta)^2+\mu}>0$, A va B o‘zgarmas kattaliklar.

(20) ni (18) dagi 1-shartga qo‘ysak, $B=0$ kelib chiqadi. Demak, (17) tenglamaning (18) shartni qanoatlantiruvchi yechimi quyidagi ko‘rinishda ekanligi kelib chiqadi:

$$\Phi(\varphi)=AF\left[(\alpha+\beta+\omega)/2,\ (\alpha+\beta-\omega)/2,1/2+\beta;\sin^2\varphi\right]\quad(21)$$

(21) ni (18) ni ikkinchi shartiga qo‘yib, hamda maxsus funksiyalar formulalaridan foydalanib, quyidagi ifodaga kelamiz:

$$\lim_{\varphi\to\frac{\pi}{2}}A(\cos\varphi)^{2\alpha+1}\frac{(\alpha+\beta)^2-\omega^2}{1+2\beta}F\left(\frac{\alpha+\beta+\omega+2}{2},\frac{\alpha+\beta-\omega+2}{2},\frac{3}{2}+\beta;\sin^2\varphi\right)=0$$

Bu yerdan,

$$A\frac{(\alpha+\beta)^2-\omega^2}{1+2\beta}F\left(\frac{\alpha+\beta+\omega+2}{2},\frac{\alpha+\beta-\omega+2}{2},\frac{3}{2}+\beta;1\right)=0$$

bo‘lishi kelib chiqadi. Endi keyingi qadamda quyidagi formulani qo‘llaymiz:

$$F(c-a,c-b,c;1)=\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)},\ c-a-b<0$$

$$F\left(\frac{\alpha+\beta+\omega+2}{2},\frac{\alpha+\beta-\omega+2}{2},\frac{3}{2}+\beta;1\right)=\frac{\Gamma\left(\frac{3}{2}+\beta\right)\Gamma\left(\alpha-\frac{1}{2}\right)}{\Gamma\left(\frac{1+\alpha-\beta+\omega}{2}\right)\Gamma\left(\frac{1+\alpha-\beta-\omega}{2}\right)}$$

$\Gamma\left[1-\left(\frac{1+\alpha-\beta+\omega}{2}\right)\right]\neq0$ deb hisoblab va $\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)}$ formuladan foydalanib, quyidagi almashtirishni bajaramiz:

$$A\frac{(\alpha+\beta)^2-\omega^2}{1+2\beta}\frac{\Gamma\left(\frac{3}{2}+\beta\right)\Gamma\left(\alpha-\frac{1}{2}\right)}{\pi\Gamma\left(\frac{1+\alpha-\beta-\omega}{2}\right)}\sin\left[\pi\left(\frac{1+\alpha-\beta+\omega}{2}\right)\right]=0$$

bu yerda $A\neq0$

$\sin\left[\pi\left(\frac{1+\alpha-\beta+\omega}{2}\right)\right]=0$, ekanligidan, $\omega_n=2n-\alpha+\beta$, $n\in N$. Demak {(17),(18)} masalaning xos sonlari $\mu_n=\omega_n^2-(\alpha+\beta)^2$, xos funksiyalari esa

$$\Phi_n(\varphi)=A_nF\left(\beta+n,\alpha-n,\frac{3}{2}+\beta;\sin^2\varphi\right) \tag{22}$$

ko‘rinishda bo‘lar ekan.

Endi {(13), (14)} masalani tadqiqiga o‘tamiz. Buning uchun (10) tenglamada o‘zgaruvchilarni $R(r)=(\rho/\sqrt{\lambda})^{1/2-(1+\alpha+\beta+\gamma)}M(\rho)$, $\rho=\sqrt{\lambda}r$ formula bo‘yicha almashtiramiz, natijada bizda Bessel tenglamasi paydo bo‘ladi:

$$\rho^2M''(\rho)+\rho M'(\rho)+(\rho^2-\sigma^2)M(\rho)=0. \tag{23}$$

Almashtirishlarni bajargandan keyin (23) tenglamaning umumiy yechimini topamiz:

$$R(r)=c_1 r^{1/2-(1+\alpha+\beta+\gamma)} J_\sigma(\sqrt{\lambda}r)+c_2 r^{1/2-(1+\alpha+\beta+\gamma)} Y_\sigma(\sqrt{\lambda}r),\ 0<r<a \quad (24)$$

Bu yerda $\sigma^2=\left(\frac{1}{2}+\alpha+\beta+\gamma\right)^2+\chi$, c_1 va c_2- o‘zgarmas kattaliklar, $J_\sigma(z)$ va $Y_\sigma(z)$ lar esa σ tartibli Besselning birinchi va ikkkinchi tur funksiyalari hisoblanadi hamda uning ko‘rinishi quyidagicha:

$$J_\sigma(z)=\sum_{n=0}^{\infty}\frac{(-1)^n (z/2)^{2n+\sigma}}{n!\,\Gamma(n+\sigma+1)}, \quad (25)$$

$$Y_\sigma(z)=\frac{J_\sigma(z)\cos\sigma\pi-J_{-\sigma}(z)}{\sin\sigma\pi},\quad \mathrm{Re}\,\sigma>0. \quad (26)$$

(24) yechimni (14) shartga bo‘ysundiramiz va (25), (26) formulalarni inobatga olib yozamiz:

$$R(r)=c_1 r^{1/2-(1+\alpha+\beta+\gamma)} J_\sigma(\sqrt{\lambda}r) \quad (27)$$

{(15), (16)} masalani tadqiq etishga kirishamiz.

Buning uchun (15) tenglamada yangi o‘zgaruvchi $\xi=\sin^2\theta$ ga o‘tib, quyidagini hosil qilamiz:

$$\xi(1-\xi)\tilde{T}''(\xi)+\left[(1+\alpha+\beta)-(\alpha+\beta+\gamma+3/2)\xi\right]\tilde{T}'(\xi)+$$

$$+(1/4)(\chi-\mu/\xi)\tilde{T}(\xi)=0,\quad 0<\xi<1, \quad (28)$$

(28) oddiy differensial tenglama bo‘lib, u Goyn tipidagi umumlashgan tenglama hisoblanadi, bu yerda $\tilde{T}(\xi)=T(\arcsin\sqrt{\theta})$.

$$T(t)=C\sum_{k=0}^{+\infty}A_k F(a_1,a_2;a_3+k;t)+$$

$$+Dt^{1-a_3-a_4}\sum_{k=0}^{+\infty}A_k F(a_1-a_3+1,a_2-a_3+1;2-a_3+k;t) \quad (29)$$

(29) formulaga asoslanib, (28) tenglamaning umumiy yechimini topamiz va kiritilgan belgilashlarni hisobga olib, (15) tenglamaning umumiy yechimini quyidagicha yozamiz:

$$T(\theta)=C\sum_{k=0}^{\infty}A_k F\left[\frac{1}{4}+\frac{\alpha+\beta+\gamma+\sigma}{2},\frac{1}{4}+\frac{\alpha+\beta+\gamma-\sigma}{2};1+\alpha+\beta+k;\sin^2\theta\right]$$

$$+D\,(\sin\theta)^{-2\alpha-2\beta}F\left[\frac{1}{4}+\frac{\gamma-\alpha-\beta+\sigma}{2},\frac{1}{4}+\frac{\gamma-\alpha-\beta-\sigma}{2};1-\alpha-\beta+k;\sin^2\theta\right]\quad(30)$$

bu yerda C, D -o‘zgarmas sonlar, $\sigma^2=(\alpha+\beta+\gamma+1/2)^2+\chi$, va $F[...]$-Gausssning gipergeometrik funksiyasi, A_k koeffitsientlar esa quyidagicha aniqlanadi:

$$A_0=1,\ A_{k+1}=\frac{k(\alpha+\beta+k)-\mu_n/4}{(k+1)(1+\alpha+\beta+k)}A_k,\ k=0,1,2,...$$

(15) tenglamaning (16) shartni birinchisini qanoatlantiruvchi yechimi, $C=1$ bo‘lganda, quyidagicha ko‘rinishga ega:

$$T(\theta)=\sum_{k=0}^{\infty}A_kF\left[\frac{1}{4}+\frac{\alpha+\beta+\gamma+\sigma}{2},\frac{1}{4}+\frac{\alpha+\beta+\gamma-\sigma}{2};1+\alpha+\beta+k;\sin^2\theta\right]=$$

$$=\sum_{k=0}^{\infty}\frac{(\alpha+\beta+n/2)_k(-n/2)_k}{k!(1+\alpha+\beta)_k}\times\quad(31)$$

$$\times F\left[\frac{1}{4}+\frac{\alpha+\beta+\gamma+\sigma}{2},\frac{1}{4}+\frac{\alpha+\beta+\gamma-\sigma}{2};1+\alpha+\beta+k;\sin^2\theta\right]$$

Rabe alomati asosida quyidagi sonli qatorning yaqinlashuvini isbotlash mumkin:

$$T(0)=\sum_{k=0}^{\infty}\frac{(\alpha+\beta+n/2)_k(-n/2)_k}{k!(1+\alpha+\beta)_k}.$$

Gipergeometrik funksiyaning yoyilmasiga asoslanib, uning yig’indisi quyidagiga teng bo‘ladi:

$$F(\alpha+\beta+n/2,-n/2;1+\alpha+\beta;1)=$$

$$=\begin{cases}\dfrac{\Gamma(1+\alpha+\beta)}{\Gamma(1+\alpha+\beta+n/2)\,\Gamma(1-n/2)},n=1,3,5,...\\ 0,n=2,4,6,...\end{cases}$$

Demak, (29) funksiya (16) shartning birinchisini qanoatlantiradi.

Endi (29) funksiyani (16) shartning ikkinchisi uchun ham ishlaymiz:

$$\lim_{\theta\to\frac{\pi}{2}}(\cos\theta)^{2\gamma}T'(\theta)=$$

$$=2\left[\left(\frac{1}{4}+\frac{\alpha+\beta+\gamma}{2}\right)^2-\frac{\sigma^2}{4}\right]\times$$

$$\times\frac{\Gamma\ 1+\alpha+\beta\ \Gamma\ 1/2+\gamma}{\Gamma\ \ 5/2+\alpha+\beta+\gamma+\sigma\ \ /2\ \Gamma\ \ 5/2+\alpha+\beta+\gamma-\sigma\ \ /2}=0\,.$$

Olingan kasrning surat va maxrajini $\Gamma\ 1-\ 5/2+\alpha+\beta+\gamma-\sigma\ \ /2\ \neq 0$ ga ko‘paytirib, oxirgi tenglikdan quyidagini hosil qilamiz:

$$2\left[\left(\frac{1}{4}+\frac{\alpha+\beta+\gamma}{2}\right)^2-\frac{\sigma^2}{4}\right]\times$$

$$\times\frac{\Gamma\ 2+\alpha+\beta\ \sin\left[\pi\ \ 5/2+\alpha+\beta+\gamma-\sigma\ \ /2\ \right]}{\pi\Gamma\ \ 5/2+\alpha+\beta+\gamma+\sigma\ \ /2\ \Gamma^{-1}\ 1-\ 5/2+\alpha+\beta+\gamma-\sigma\ \ /2}$$

Mazkur tenglik $\sin\left[\pi\ \ 5/2+\alpha+\beta+\gamma-\sigma\ \ /2\ \right]=0$ bo‘lganda bajariladi. Bu esa trigonometrik tenglama bo‘lib, u faqat haqiqiy ildizlarga ega.

Ushbu tenglamaning yechimini ifodalovchi formuladan va $\sigma\geq 1/2+\alpha+\beta+\gamma$ tengsizlikdan hamda quyidagi:

$$5/4+\alpha/2+\beta/2+\gamma/2+\sigma/2\neq 0,-1,-2...,,$$

$$-1/4-\alpha/2-\beta/2-\gamma/2+\sigma/2\neq 0,-1,-2...,$$

munosabatdan foydalanib,

$$\sigma=\sigma_l=2l+1/2+\alpha+\beta+\gamma\,,\ l\in N \qquad (32)$$

ni hosil qilamiz.

$\sigma=\sqrt{\ 1/2+\alpha+\beta+\gamma\ ^2+\chi}$ ekanligidan foydalanib, $\chi_l=\sigma_l^2-\ 1/2+\alpha+\beta+\gamma\ ^2$, $l\in N$ ni hosil qilamiz, bu yerda σ_l - (32) tenglik bilan aniqlanadigan son. χ_l esa {(15), (16)} masalaning xos sonlari.

(29) da $\sigma=\sigma_l$, $l\in N$ deb olib, {(15), (16)} masalaning xos funksiyalarini xos soni χ_l uchun topamiz:

$$T_{nl}\ \theta\ =\sum_{k=0}^{\infty}\frac{\alpha+\beta+n/2\ _k\ \ -n/2\ _k}{k!\ 1+\alpha+\beta\ _k}\times$$

$$\times F\left(l+1/2+\alpha+\beta+\gamma,-l;1+\alpha+\beta+k;\sin^2\theta\right),\theta\in\left(0,\pi/2\right),n,l\in N \quad (33)$$

Har qanday k, l va $\theta\in\left[0,\pi/2\right]$ qiymatlar uchun $F\left(l+1/2+\alpha+\beta+\gamma,-l;1+\alpha+\beta+k;\sin^2\theta\right)$ funksiya chegaralangan bo‘ladi. Shuning uchun (33) qator $\theta\in\left(0,\pi/2\right)$ da absolut va tekis yaqinlashuvchi bo‘ladi. Natijada $T_{nl}\left(\theta\right)$ funksiya $\theta\to\pi/2$ da ham chegaralangan bo‘ladi. Yuqoridagilardan kelib chiqib, (33) qator $\left[0,\pi/2\right]$ kesmada absolut va tekis yaqinlashuvchi degan xulosaga kelish mumkin.

$T_{nl}\left(\theta\right)$ funksiyani quyidagicha yozish mumkin:

$$T_{nl}\left(\theta\right)=F_3\left(l+1/2+\alpha+\beta+\gamma,\alpha+\beta+n/2,-l,-n/2;1+\alpha+\beta;\sin^2\theta,1\right)$$
$$n,l\in N \quad (34)$$

bu yerda $F_3=\left(a,a',b,b';c;w,z\right)$, $|w|,|z|<1$ -Appelning gipergeometrik funksiyasi.

Endi (32) tenglik bilan aniqlangan $\sigma=\sigma_l=2l+1/2+\alpha+\beta+\gamma$, $l\in N$ sonlar uchun λ parametrning qiymatlarini topamiz. (27) funksiyani (14) shartning ikkinchisiga qo‘yib,

$$J_{\sigma+1}\left(\sqrt{\lambda}a\right)=0,\ \sigma=1/2+\alpha+\beta+\gamma \quad (35)$$

tenglikni hosil qilamiz.

Ma’lumki, $\nu>-1$ bo‘lganda, $J_\nu\left(x\right)$ Bessel funksiyasi chekli sondagi nol nuqtalarga ega bo‘lib, ularning barchasi haqiqiy va juft-juft qarama qarshi ishoralidir. Shu ma’lumotga asosan va $\sigma+1>0$ ekanligidan, (35) tenglama ildizlari chekli sonda bo‘ladi. (35) tenglamaning musbat nollarini δ_{ml} bilan belgilab, {(13), (14)} masalaning trivial bo‘lmagan yechimlari mavjud bo‘ladigan λ parametr qiymatlarini , ya’ni {(13), (14)} masalaning xos qiymatlarini topamiz: $\lambda=\lambda_{ml}=\left(\frac{\delta_{ml}}{a}\right)^2$, $m,l\in N$. Unga mos xos funksiyalar esa,

$$R_{ml}\left(r\right)=c_{ml}r^{-\left(1/2+\alpha+\beta+\gamma\right)}J_{\sigma_l}\left(\frac{\delta_{ml}}{a}r\right),\ m,l\in N \quad (36)$$

ko‘rinishda bo‘ladi.

Demak, biz qo‘ygan Neyman masalasi sanoqli sondagi xos qiymat va xos funksiyalarga ega. Uning xos qiymatlari $\lambda = \lambda_{ml} = \left(\frac{\delta_{ml}}{a}\right)^2$, $m,l \in N$ sonlaridan iborat bo‘lib, xos funksiyalari esa (22), (34), (36)formulalarga asosan quyidagi tenglik bilan aniqlanadi:

$$u_{nml} = c_{nml} r^{-1/2+\alpha+\beta+\gamma} J_{\sigma_l}\left(\frac{\delta_{ml}}{a} r\right) F\left(\beta+n, \alpha-n, \frac{3}{2}+\beta; \sin^2\varphi\right) \times$$

$$\times F_3 \; l+1/2+\alpha+\beta+\gamma, \alpha+\beta+n/2, -l, -n/2; 1+\alpha+\beta; \sin^2\theta, 1$$

bu yerda, c_{nml} -o‘zgarmas son.

XULOSA. Ushbu maqolada markazi koordinata boshida bo‘lgan sharning sakkizdan bir qismida uch o‘lchovli fazoda uchta singulyar koeffitsientli elliptik tipdagi tenglama uchun Neyman masalasi tadqiq qilindi. Tadqiqotlar natijasi shuni ko‘rsatadiki, bunday tenglamalar sharsimon sohada nisbatan kam o‘rganilgan bo‘lib, keltirilgan metodlar yangi yechimlar olish imkonini beradi. Bu natijalar kelgusida uch o‘lchovli elliptik tenglamalarni va ularning turli cheklangan sohalardagi vazifalarini yechishda qo‘llanishi mumkin.

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