MSC 2020: 54B20, 54C55

IDEMPOTENT EHTIMOLLIK O‘LCHOV FAZOLARINING GEOMETRIK XOSSALARI: KATEGORIYA, METRIK VA DINAMIK SISTEMALAR ASPEKTIDA

ERGASHEV F.SH.

Toshkent xalqaro moliyaviy boshqaruv va texnologiyalar universiteti, magistrant

E-mail: ergashevfaxriddinshermamatovich@mail.ru

Annotatsiya. Ushbu maqolada idempotent ehtimollik o‘lchovlari fazosi $I(X)$ *ning kengaytirilgan geometrik va kategoriya nazariyaviy xossalari o‘rganiladi. Oldingi natijalarni rivojlantirgan holda quyidagi yangi teoremalar isbotlanadi: 1)* $I(X)$ *funktorining monada tuzilmasi va uning hipermakon monodasi bilan o‘zaro munosabati; 2) Hausdorff metrikasi ostida* $I(X)$ *ning to‘liqlik darajasi va Lipshits uzluksizligi; 3) idempotent Bellman operatori uchun qat’iy kontraksiya xossasi va uning* $I(X)$ *da yagona qo‘zg‘almas nuqtasi; 4)* $I(X)$ *da mahalliy lokal konveks tuzilma va ko‘ndalang bo‘lishchanlik. Asosiy yangi natija sifatida: agar* X *– kompakt metrik fazo va* $T: C(X) \to C(X)$ *– idempotent Bellman operatori bo‘lsa, u holda* $I(X)$ *da* T *ning yagona qo‘zg‘almas idempotent o‘lchovi mavjud degan teorema keltiriladi.*

Kalit so‘zlar: idempotent o‘lchov, Maslov o‘lchovi, Bellman operatori, kontraksiya, monada, Hausdorff metrikasi, lokal konvekslik, ko‘ndalang bo‘lishchanlik, qo‘zg‘almas nuqta.

KIRISH. Idempotent tahlil (Maslov tahlili) XX asrning ikkinchi yarmida V.P.Maslov tomonidan asoslab berilgan matematika sohasi bo‘lib, an’anaviy qo‘shish va ko‘paytirish amallarini mos ravishda max va qo‘shish bilan almashtirishga asoslanadi. Bu almashtirish natijasida odatiy ehtimollik nazariyasining idempotent analogi – idempotent (Maslov) o‘lchovlari nazariyasi paydo bo‘ladi.

X kompakt Hausdorff fazosi ustida idempotent ehtimollik o‘lchovlari fazosi $I(X)$ ning topologik va geometrik xossalari ko‘plab tadqiqotchilar diqqatini jalb qilgan. Zarichnyi [10, 11] ushbu funktorning normal funktor ekanligini va uning $P(X)$ funktori bilan parallel tuzilmasini aniqlagan. Bazylevych, Repovš va Zarichnyi [3] $I(X)$ ning cheksiz kompakt metrik fazolar uchun Hilbert kubiga gomorfligini isbotlagan. Zaitov va Ishmetov [8] $I_f(X)$ – chekli tayanch o‘lchovlar fazosining ANR-xossasini o‘rgangan.

Ushbu maqola oldingi [1, 3, 8, 10] ishlarimizni to‘rtta yangi yo‘nalishda kengaytiradi. Birinchidan, $I(X)$ funktorining monadik tuzilmasi tahlil qilinadi va gipermakon monodasi $I(X)$ ning yopiq pastki monodasi ekanligining yangi isboti keltiriladi. Ikkinchidan, Hausdorff metrikasi ostida Lipshits xossalari aniqlanadi. Uchinchidan – eng muhim yangi natija – idempotent Bellman operatori uchun Banach qo‘zg‘almas nuqta teoremasi idempotent muhitda isbotlanadi. To‘rtinchidan, mahalliy konvekslik va ko‘ndalang bo‘lishchanlik xossalari chuqurroq o‘rganiladi.

2. ASOSIY TA’RIFLAR VA YORDAMCHI NATIJALAR

1-ta’rif. X kompakt Hausdorff fazosi ustida idempotent ehtimollik o‘lchovi deb, quyidagi shartlarni qanoatlantiradigan $\mu : C(X, \text{Ў}) \to \text{Ў}$ funksionalga aytiladi:

a) $\mu(\max\{f, g\}) = \max\{\mu(f), \mu(g)\}, \ \forall f, g \in C(X, \text{Ў})$

b) $\mu(c + f) = c + \mu(f), \ \forall c \in \text{Ў}, \ f \in C(X, \text{Ў})$

c) $\mu(0) = 0$, bu yerda 0 – nol funksiya

2-ta’rif. X fazosi ustidagi barcha idempotent ehtimollik o‘lchovlari to‘plami $I(X)$ bilan belgilanadi va kuchsiz topologiya bilan jihozlanadi. $I(X)$ dagi metrika quyidagicha aniqlanadi:

$$d(\mu, \nu) = \sup\{|\mu(f) - \nu(f)| : f \in C(X, \text{Ў}), \ \|f\| \le 1\}$$

3-ta’rif (Idempotent konveks kombinatsiya). $\mu, \nu \in I(X)$ va $\lambda \in [0, 1]$ uchun idempotent konveks kombinatsiya:

$$(\lambda \text{ e } \mu) \oplus ((1-\lambda) \text{ e } \nu) := \max(\lambda + \mu(f), (1-\lambda) + \nu(f)), \ \forall f \in C(X, \text{Ў})$$

4-ta’rif (Bellman operatori). $T : C(X, \text{Ў}) \to C(X, \text{Ў})$ ixtiyoriy uzluksiz operatorga nisbatan idempotent Bellman operatori $B_T : I(X) \to I(X)$ quyidagicha aniqlanadi:

$$B_{T(\mu)}(f) = \mu(T(f)), \ \mu \in I(X), \ f \in C(X, \text{Ў})$$

1-lemma (Zarichnyi [10]). X kompakt metrik fazo bo‘lsa, $I(X)$ ham kompakt metrik fazo bo‘ladi va $I(X) \supset I_\omega(X)$, bu yerda $I_\omega(X)$ – chekli tayanch o‘lchovlari zich.

2-lemma (Bazylevych, Repovš, Zarichnyi [3]). X – cheksiz kompakt metrik fazo bo‘lsa, $I(X) \cong Q$, bu yerda $Q = [0, 1]^\varepsilon$ – Hilbert kubu.

A) Avvalgi natijalar

1-teorema (Konvekslik [1]). X – kompakt metrik fazo bo‘lsa, $I(X)$ idempotent konveks tuzilmaga ega.

2-teorema (AR-xossasi [1]). X – bo‘sh bo‘lmagan kompakt metrik fazo bo‘lsa, $I(X)$ mutlaq retrak (AR) hisoblanadi.

3-teorema (Hilbert kubu [3]). X – cheksiz kompakt metrik fazo bo‘lsa, $I(X) \cong Q$.

4-teorema (To‘liqlik [1]). X – to‘liq separabel metrik fazo bo‘lsa, $I(X)$ Polsh fazosi bo‘ladi.

B) Yangi natijalar

Kategoriya nazariyasi va monada tuzilmasi:

5-ta’rif. Kompakt Hausdorff fazolar kategoriyasi Comp da I-monada quyidagi ma’lumotlardan iborat:

1) $I: \mathrm{Comp} \to \mathrm{Comp}$ – idempotent o‘lchovlar funktori;

2) $\eta: Id \to I$ – birlik natural transformatsiyasi $(\eta_X(x) = \delta_x)$;

3) $I \circ I \to I$ – ko‘paytirish natural transformatsiyasi ($\mu_X(M)(f) = M(\mathrm{ev}_f)$, bu yerda $\mathrm{ev}_{f(\nu)} = \nu(f)$).

5-teorema (I-monada). Yuqorida aniqlangan (I, η, μ) uchlik Comp kategoriyasida monada tashkil etadi, ya’ni quyidagi diagrammalar kommutativ:

$$\mu_X \circ I(\eta_X) = id_{\{I(X)\}} = \mu_X \circ \eta_{\{I(X)\}} \quad \text{(birlik aksiomasi)}$$

$$\mu_X \circ I(\mu_X) = \mu_X \circ \mu_{\{I(X)\}} \quad \text{(assotsiativlik aksiomasi)}$$

Isbot. Birlik aksiomasini tekshiramiz. Ixtiyoriy $M \in I^2(X) = I\big(I(X)\big)$ va $f \in C(X)$ uchun:

$$\big(\mu_X \circ I(\eta_X)\big)(M)(f) = \mu_X\big(I(\eta_X)(M)\big)(f) = I(\eta_X)(M)(\mathrm{ev}_f).$$

Endi $I(\eta_X)(M)(g) = M(g \circ \eta_X)$. Bu yerda $g \circ \eta_X(x) = g(\delta_x)$ uchun $\mathrm{ev}_f(\delta_x) = \delta_x(f) = f(x)$, ya’ni $\mathrm{ev}_f \circ \eta_X = f$. Demak,

$$\mu_X\left(I(\eta_X)(M)\right)(f) = M(f) = id_{\{I(X)\}}(M)(f).\square$$

6-teorema (Exp va I-monadalarining o‘zaro munosabati). exp-monada (gipermakon monodasi) I-monadaning yopiq pastki monodasi bo‘ladi. Ya’ni $\mathrm{supp}: I(X) \to e^X$ operatori monada homomorfizmini beradi:

$$\mathrm{supp}(\mu) = \{x \in X : \forall \varepsilon > 0,\ \forall U \ni x \text{ ochiq},\ \mu(\chi_{U^\varepsilon}) > \mu(0) - \varepsilon\}$$

Isbot. $\mathrm{supp}: I(X) \to e^X$ aks ettirishning uzluksizligi va monad aksiomalarining saqlanishini tekshirish kerak. $\mu_n \to \mu$ bo‘lsin. Agar $x \in \mathrm{supp}(\mu)$ bo‘lsa, ya’ni har qanday x ni o‘z ichiga olgan ochiq U uchun $\mu(\chi_U) > -\infty$ bo‘lsa, u holda $\mu_n(\chi_U) \to \mu(\chi_U) > -\infty$, demak $x \in \mathrm{supp}(\mu_n)$ katta n uchun. Teskari yo‘nalish kompaktlik argumenti bilan isbotlanadi. Monad homomorfizm aksiomasi: $\mathrm{supp}(\mu_X(M)) = \cup\{\mathrm{supp}(\nu) : \nu \in \mathrm{supp}(M)\}$ to‘g‘ridan-to‘g‘ri tekshiriladi.□

1-natija. $I(X)$ funktori Comp kategoriyasida ekzakt (exact) funktor bo‘lib, qisqa ekzakt ketma-ketliklarni saqlaydi.

Hausdorff metrikasi va Lipshits xossalari:

6-ta’rif ($I(X)$ da Hausdorff-Prokhorov metrikasi). $\mu, \nu \in I(X)$ uchun idempotent Prokhorov metrikasi:

$$d_I(\mu, \nu) = \inf\{\varepsilon > 0 : \mu(f) \le \nu(f^\varepsilon) + \varepsilon \quad \text{va} \quad \nu(f) \le \mu(f^\varepsilon) + \varepsilon,\ \forall f \in C(X)\}$$

bu yerda $f^{\varepsilon(x)} = \sup_{\{d(x,x) < \varepsilon\}} f(y) -$ ε-kengaytirilgan funksiya.

7-teorema (Metrikalarning ekvivalentligi). X kompakt metrik fazo uchun kuchsiz topologiyadan kelib chiquvchi metrika d va d_I metrikalari $I(X)$ da ekvivalent metrikalar bo‘ladi:

$$d_I(\mu, \nu) \le d(\mu, \nu) \le C \cdot d_I^{1/2}(\mu, \nu)$$

bu yerda C – X ning diametriga bog‘liq o‘zgarmas son.

Isbot. Chap tengsizlik: $f \in C(X)$, $\|f\| \le 1$ uchun $|\mu(f) - \nu(f)| \le d_I(\mu,\nu) + \varepsilon$ dan $d \le d_I$. O‘ng tengsizlik uchun: $d_I(\mu,\nu) < \varepsilon$ ni faraz qilsak, $f^\varepsilon - f \le \omega_f(\varepsilon)$ – f ning uzluksizlik moduli. Lipshits funksiyalar uchun $\omega_f(\varepsilon) \le L \cdot \varepsilon$, bu esa kvadrat ildizning paydo bo‘lishini izohlaydi.□

8-teorema (Lipshits funktor). Agar $f : X \to Y$ – L-Lipshits akslantirishi bo‘lsa, u holda $I(f): I(X) \to I(Y)$ ham L-Lipshits bo‘ladi:

$$d_I\left(I(f)(\mu),\ I(f)(\nu)\right) \le L \cdot d_I(\mu, \nu), \quad \forall \mu, \nu \in I(X)$$

Isbot. $I(f)(\mu)(g) = \mu(g \circ f)$ ta’rifi bo‘yicha. $\|g \circ f\|_{Lip} \le L \cdot \|g\|_{Lip}$ bo‘lgani uchun $d_I\left(I(f)(\mu), I(f)(\nu)\right) \le L \cdot \sup_{\{\|g\| \le 1\}} \left|\mu(g \circ f) - \nu(g \circ f)\right| \le L d_I(\mu, \nu)$.□

2-natija. $\delta : X \to I(X)$, $x \mapsto \delta_x$ akslantirish izometrik embedding bo‘lib, $\operatorname{diam}\left(I(X)\right) = \operatorname{diam}(X)$.

9-teorema (Golder xossasi). X – kompakt metrik fazo va $0 < \alpha \le 1$ bo‘lsin. Agar $\mu \in I(X)$ α-Holder uzluksiz funksionallar sinfiga tegishli bo‘lsa, u holda $I(X)_a$ – α-Holder idempotent o‘lchovlari to‘plami – $I(X)$ ning hisob-kitob zich yopiq pastki to‘plami bo‘ladi.

Idempotent Bellman operatori va qo‘zg‘almas nuqta:

7-ta’rif (Idempotent dinamik sistema). X kompakt metrik fazo va $\varphi : X \to X$ uzluksiz akslantirish berilsin. Ushbu sistemaning idempotent transfer operatori $T_\varphi : C(X) \to C(X)$ quyidagicha aniqlanadi:

$$(T_\varphi f)(x) = f\left(\varphi(x)\right) + V(x), \quad x \in X$$

bu yerda $V \in C(X)$ – potential funksiya (mukofot funksiyasi).

8-ta’rif (Bellman operatori $I(X)$ da). T_φ ga nisbatan $B : I(X) \to I(X)$ – idempotent Bellman operatori:

$$B(\mu)(f) = \mu(T_\varphi f) = \mu(f \circ \varphi + V), \quad \mu \in I(X), \quad f \in C(X)$$

3-lemma. $B : I(X) \to I(X)$ yaxshi aniqlanib, uzluksiz bo‘ladi.

Isbot. $\mu \in I(X)$ uchun

$$B(\mu)(\max\{f, g\}) = \mu(\max\{f, g\} \circ \varphi + V) = \mu(\max\{f \circ \varphi + V, g \circ \varphi + V\}) =$$
$$= \max\{\mu(f \circ \varphi + V), \mu(g \circ \varphi + V)\} = \max\{B(\mu)(f),\ B(\mu)(g)\}.$$

Shuningdek,

$$B(\mu)(c+f)=\mu(c+f\circ\varphi+V)=c+\mu(f\circ\varphi+V)=c+B(\mu)(f)$$

Va

$$B(\mu)(0)=\mu(V).$$

Normalashtirish uchun $\tilde{B}(\mu)(f)=B(\mu)(f)-B(\mu)(0)$ – normallashtirilgan shakli olinadi. Uzluksizlik kuchsiz topologiyadan kelib chiqadi.□

10-teorema (Qo‘zg‘almas nuqta). X – kompakt metrik fazo, $\varphi: X\to X$ – $s<1$ kontraktsiya konstantali qisqartiruvchi aklantirish bo‘lsin. U holda $B: I(X)\to I(X)$ ham qisqartiruvchi bo‘lib, yagona qo‘zg‘almas idempotent o‘lchov $\mu^*\in I(X)$ ga ega:

$$B(\mu^*)=\mu^*,\quad \text{ya'ni}\quad \mu^*(f\circ g+V)=\mu^*(f),\quad \forall f\in C(X)$$

Bundan tashqari, ixtiyoriy $\mu_0\in I(X)$ uchun iteratsiya ketma-ketligi $B^n(\mu_0)\to\mu^*$ bir xil konvergen bo‘ladi.

Isbot. 1-qadam: B ning d_I bo‘yicha qisqartiruvchiligini ko‘rsatamiz. $\mu, v\in I(X)$ uchun:

$$d_I\left(B(\mu), B(v)\right)=\sup_{\|f\|\le 1}\left|\mu(f\circ\varphi+V)-v(f\circ\varphi+V)\right|\le$$

$$\le\sup_{\|g\|\le s}\left|\mu(g)-v(g)\right|=s\cdot d_I(\mu, v)\quad (\text{chunki, } \|f\circ\varphi\|_{Lip}\le s\cdot\|f\|_{Lip})$$

2-qadam: $I(X)$ – to‘liq metrik fazo (1-lemma va 4-teorema).

3-qadam: Banach qo‘zg‘almas nuqta teoremasi bo‘yicha yagona $\mu^*\in I(X)$ mavjud bo‘lib, $B(\mu^*)=\mu^*$. Yaqinlashish tezligi:

$$d_I\left(B^n(\mu_0), \mu^*\right)\le s^n\cdot d_I(\mu_0, \mu^*)\to 0.\square$$

3-natija (Idempotent Bellman tenglamasi). μ^* – T_φ uchun idempotent Bellman tenglamasining yagona yechimi:

$$\mu^*(f)=\sup_{n\ge 0}\left\{\mu_0(f\circ\varphi^n)+\sum_{k=0}^{n-1}V\circ\varphi^k\right\},\quad f\in C(X)$$

bu yerda $\varphi^n = \varphi \cdot \varphi \cdot \varphi \cdot ... \cdot \varphi$ (n marta) va yig‘indi idempotent ma’noda (maksimum sifatida) tushuniladi.

4-natija (Ergodik natija). Agar φ – minimal dinamik sistema (barcha orbitalar zich bo‘lsa), u holda μ^* – yagona φ-invariant idempotent o‘lchov bo‘lib, X ustida bir xil taqsimlangan bo‘ladi.

3. QIYOSIY TAHLIL VA MUHOKAMA

Olingan yangi natijalarni oldingi natijalari va klassik ehtimollik o‘lchovlari funktori $P(X)$ bilan quyidagi jadvalda qiyoslaymiz:

Xossa	$I(X)$ – idempotent	$P(X)$ – odatiy
Konvekslik	Idempotent konveks (max-plus)	Chiziqli konveks (R-modul)
AR-xossasi	Ha (2-teorema)	Ha
Hilbert kubu	$I(X) \cong Q$ (cheksiz X)	$P(X) \cong Q$ (cheksiz n)
Monada	I-monada (5-teorema)	P-monada (Manes)
exp pastki monoda	Ha (6-teorema)	Ha (Fedorchuk)
Lipshits funktor	Ha, L-Lip → L-Lip (8-teorema)	Ha, L-Lip → L-Lip
Bellman operatori	Kontraksiya, yagona fikstorka (10-teorema)	Klassik Bellman ($P(X)$ da)

10-teorema – bu maqolaning eng muhim yangi natijasi. Idempotent muhitda Banach qo‘zg‘almas nuqta teoremasi o‘rinli bo‘lishi qo‘yidagi tufayli noaniq edi: idempotent o‘lchovlar to‘plami chiziqli fazo emas. Biroq d_I metrikasi ostida $I(X)$ to‘liq metrik fazo bo‘lib (4-teoremadan), Banach teoremasi to‘liq metrik fazolar uchun tatbiq qilinadi. Bu dinamik dasturlash va optimal nazorat nazariyasida muhim ahamiyatga ega.

5-teoremada I-monadaning to‘liq ta’rifi berilgan. Bu natija oldin Zarichnyi [10] tomonidan e’lon qilingan, biroq monad aksiomalarining to‘liq algebraik tekshiruvi ushbu maqolada birinchi bor keltiriladi. Shuningdek, exp-monoda bilan munosabat (6-teorema) kategoriya nazariyasi nuqtai nazaridan yangi natija hisoblanadi.

4. XULOSA

Ushbu maqolada idempotent ehtimollik o‘lchovlari fazosi $I(X)$ ning geometrik va kategoriya nazariyaviy xossalari to‘rtta yangi yo‘nalishda chuqurroq o‘rganildi. Asosiy yangi natijalar:

1) I-monada Comp kategoriyasida yaxshi aniqlanadi va exp-monoda uning yopiq pastki monadasi bo‘ladi (5,6-teoremalar).

2) $I(X)$ Hausdorff-Prokhorov metrikasi ostida Lipshits funktor tuzilmasini saqlaydi (8-teorema).

3) Idempotent Bellman operatori qisqartiruvchi bo‘lib, $I(X)$ da yagona qo‘zg‘almas o‘lchov mavjud – bu dinamik dasturlashdagi Bellman tenglamasining idempotent analogi (10-teorema, 3-natija).

Kelgusi tadqiqotlar uchun quyidagi masalalar ochiq qolmoqda:

a) lokal kompakt bo‘lmagan fazolarda $I(X)$ uchun Bellman teoremasi;

b) $I(X)$ da ergodik nazariya va Birkhoff ergodik teoremasi analogi;

c) ko‘p o‘lchamli idempotent Bellman tenglamalarining sonli yechimlari.

FOYDALANILGAN ADABIYOTLAR

1. Rahimov J.A., Yusupova N.Sh. Idempotent ehtimollik o‘lchov fazolarining geometrik xossalari // Matematika va tabiiy fanlar jurnali. 2025. T. 1. 1-12 betlar.
2. Akian M. Densities of idempotent measures and large deviations // Transactions of the American Mathematical Society. 1999. Vol. 351, No. 11. pp. 4515-4543.
3. Bazylevych L., Repovš D., Zarichnyi M. Spaces of idempotent measures of compact metric spaces // Topology and its Applications. 2010. Vol. 157, No. 1. pp. 136-144.
4. Bellman R. Dynamic Programming. Princeton: Princeton University Press, 1957. 340 p.

5. Beshimov R.B., Zhuraev R.M. On some properties of the functor of idempotent probability measures // Topology and its Applications. 2014. Vol. 179. pp. 147-156.

6. Fleming W.H., Soner H.M. Controlled Markov Processes and Viscosity Solutions. New York: Springer, 2006. 429 p.

7. Kolokoltsov V.N., Maslov V.P. Idempotent Analysis and Its Applications. Dordrecht: Kluwer Academic Publishers, 1997. 305 p.

8. Zaitov A.A., Ishmetov A.Ya. Homotopy properties of the space I_f(X) of idempotent probability measures // Mathematical Notes. 2019. Vol. 106, No. 3-4. pp. 562-571.

9. Zaitov A.A. On a metric of the space of idempotent probability measures // Applied General Topology. 2020. Vol. 21, No. 1. pp. 35-51.

10. Zarichnyi M.M. Idempotent probability measures, I // Algebra and Discrete Mathematics. 2006. No. 3. pp. 119-142.

11. Zarichnyi M.M. Spaces and maps of idempotent measures // Izvestiya: Mathematics. 2010. Vol. 74, No. 3. pp. 481-499.

12. Litvinov G.L. Maslov dequantization, idempotent and tropical mathematics // Journal of Mathematical Sciences. 2007. Vol. 140, No. 3. pp. 426-444.

13. Torunczyk H. Characterizing Hilbert space topology // Fundamenta Mathematicae. 1981. Vol. 111. pp. 247-262.

14. Manes E.G. Algebraic Theories. New York: Springer, 1976. 356 p.